Real Analysis MAA 6616
Lecture 21
Functions of Bounded Variation

Let $f : [a,\ b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function. For a partition $P = \{x_0, \cdots, x_n\}$ of the interval $[a,\ b]$ (i.e. $x_0 = a < x_1 < x_2 < \cdots < x_n = b$) we associate the variation of f with respect to the partition P: $S(f, P) = \sum_{j=1}^{n} \left| f(x_j) - f(x_{j-1}) \right|$.

The total variation of f on $[a,\ b]$ (or simply the variation) is

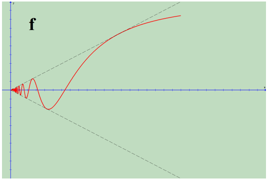
$$V(f) = V(f, [a,\ b]) = \sup\{S(f, P) :\ P \text{ partition of } [a,\ b]\}$$

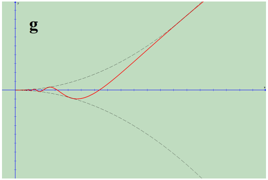
f is said to be of bounded variation on $[a,\ b]$ if $V(f, [a,\ b]) < \infty$. Denote by $\mathrm{BV}[a,\ b]$ the space of functions of bounded variations.

- If f is monotone on $[a,\ b]$, then $f \in \mathrm{BV}[a,\ b]$ and $V(f) = |f(b) - f(a)|$.
- If f is Lipschitz on $[a,\ b]$ with Lipschitz constant c, then $f \in \mathrm{BV}[a,\ b]$ and $V(f) \leq c(b - a)$. Indeed, for any partition P we have
$$S(f, P) = \sum_{j=1}^{n} \left| f(x_j) - f(x_{j-1}) \right| \leq \sum_{j=1}^{n} c \left| x_j - x_{j-1} \right| = c(b - a)$$
- If $f \in \mathrm{BV}[a,\ b]$, then f is bounded: Let $x \in [a,\ b]$, and consider the partition $P = \{a,\ x,\ b\}$. Then
$$|f(x)| - |f(a)| \leq S(f, P) = |f(x) - f(a)| + |f(b) - f(x)| \leq V(f)$$
Hence $|f(x)| \leq V(f) + |f(a)|$ for all x.
- If $c \in (a,\ b)$ then $V(f, [a,\ b]) = V(f, [a,\ c]) + V(f, [c,\ b])$. It follows that for every $a \leq u < v \leq b$, we have $V(f, [u,\ v]) = V(f, [a,\ v]) - V(f, [a,\ u])$

Examples

1. The Dirichlet function $\chi_{\mathbb{Q}} \notin \mathrm{BV}[0,\ 1]$: Let $N \in \mathbb{N}$. Consider a partition P with N points of $[0,\ 1]$ such that $x_j \in \mathbb{Q}$ if j is odd and $x_j \notin \mathbb{Q}$ if j is even so that $\left|\chi_{\mathbb{Q}}(x_j) - \chi_{\mathbb{Q}}(x_{j-1})\right| = 1$. Then $S(\chi_{\mathbb{Q}}, P) = N$. Since N is arbitrary, then $V(\chi_{\mathbb{Q}}) = \infty$.
2. Consider the functions $f(x) = x \sin \dfrac{\pi}{x}$ and $g(x) = x^2 \sin \dfrac{\pi}{x}$. Then $f \notin \mathrm{BV}[0,\ 1]$ and $g \in \mathrm{BV}[0,\ 1]$





- Let $N \in \mathbb{N}$ and consider the partition P of $[0,\ 1]$ with N points given by $x_j = 1/(N - j + 1)$ for j even and $x_j = 1/(N - j + 1 + 1/2)$ for j odd so that $\sin(\pi/x_j) = 0$ if j even and $\sin(\pi/x_j) = \pm 1$ if j is odd. Then
$$S(f, P) = \sum_{j=1}^{N} \frac{1}{N + 1 - j + \frac{1}{2}} = \sum_{k=1}^{N} \frac{1}{k + \frac{1}{2}} \longrightarrow \infty \text{ as } N \to \infty$$
Therefore $V(f, [0,\ 1]) = \infty$.
- Note that g is differentiable with $g'(x) = 2x \sin \dfrac{\pi}{x} - \pi \cos \dfrac{\pi}{x}$ and $|g'(x)| \le 2 + \pi$. we can apply the MTV on any interval $[c,\ d] \subset [0,\ 1]$ to find $\alpha \in (c,\ d)$ so that $|g(d) - g(c)| = |g'(\alpha)(d - c)| \le (2 + \pi)(d - c)$. Hence if $P = \{x_0, \cdots, x_n\}$ is any partition of $[0,\ 1]$, then
$$S(g, P) = \sum_{j=1}^{n} |g(x_j) - g(x_{j-1})| \le \sum_{j=1}^{n} (2 + \pi)(x_j - x_{j-1}) = (2 + \pi)$$
Hence $V(g, [0,\ 1]) \le 2 + \pi$ and $g \in \mathrm{BV}[0,\ 1]$.

Theorem (1)

Let $f, g \in \mathrm{BV}[a,\ b]$. Then the functions $\alpha f + \beta g$ (with $\alpha, \beta \in \mathbb{R}$) and fg are also with bounded variations. If there exists $\epsilon > 0$ such that $|g| > \epsilon$ in $[a,\ b]$, then f/g is also of bounded variation.

Proof.

Suppose that f and g are functions with bounded variations. Let P be a partition of $[a,\ b]$. It follows from the triangle inequality that

$$S(f+g, P) \leq S(f, P) + S(g, P) \leq V(f, [a,\ b]) + V(g, [a,\ b])$$

Therefore $V(f+g) \leq V(f) + V(g)$ and so $f + g \in \mathrm{BV}[a,\ b]$ If $\alpha \in \mathbb{R}$, then $S(\alpha f, P) = |\alpha| S(f, P) \leq |\alpha| V(f)$ and so αf is with bounded variation.

Since $f, g \in \mathrm{BV}[a,\ b]$, then there exist $A > 0$ and $B > 0$ such that $|f| \leq A$ and $|g| \leq B$. We have

$$\begin{aligned} \left|f(x_j)g(x_j) - f(x_{j-1})g(x_{j-1})\right| &= \left|f(x_j)g(x_j) - f(x_{j-1})g(x_j) + f(x_{j-1})g(x_j) - f(x_{j-1})g(x_{j-1})\right| \\ &\leq \left|g(x_j)\right| \left|f(x_j) - f(x_{j-1})\right| + \left|f(x_{j-1})\right| \left|g(x_j) - g(x_{j-1})\right| \\ &\leq B \left|f(x_j) - f(x_{j-1})\right| + A \left|g(x_j) - g(x_{j-1})\right| \end{aligned}$$

This implies that

$$S(fg, P) \leq B\, S(f, P) + A\, S(g, P) \leq B\, V(f) + A\, V(g)$$

and consequently $V(fg) \leq B\, V(f) + A\, V(g)$.

Suppose that $|g| > \epsilon$, then

$$\left|\frac{1}{g(x_j)} - \frac{1}{g(x_{j-1})}\right| = \frac{\left|g(x_{j-1}) - g(x_j)\right|}{\left|g(x_j)g(x_{j-1})\right|} \leq \frac{1}{\epsilon^2}\ \left|g(x_j)g(x_{j-1})\right|.$$

Thus

$$S(\frac{1}{g}, P) \leq \frac{1}{\epsilon^2}\ S(g, P) \leq \frac{1}{\epsilon^2}\ V(g)$$

Hence $1/g$ has bounded variation. □

Theorem (2)

A function f is of bounded variation on $[a,\ b]$ if and only if f can be written as the difference of two increasing functions on $[a,\ b]$:

$$f \in \mathrm{BV}[a,\ b] \iff f = g - h \text{ where } g, h : [a,\ b] \longrightarrow \mathbb{R} \text{ increasing}$$

Proof.

"$\Longrightarrow$" Let $f \in \mathrm{BV}[a,\ b]$. Define the function h on $[a,\ b]$ by $h(x) = V(f,\ [a,\ x])$. The function h is increasing. Let $g = f + h$. We remains to verify that g is increasing. Let $a \leq x_1 < x_2 \leq b$. Then

$$f(x_1) - f(x_2) \leq |f(x_1) - f(x_2)| \leq V(f,\ [x_1,\ x_2]) = V(f,\ [a,\ x_2]) - V(f,\ [a, x_1]) = h(x_2) - h(x_1).$$

It follows that $g(x_1) = f(x_1) + h(x_1) \leq g(x_2) = f(x_2) + h(x_2)$.

"$\Longleftarrow$" If $g, h : [a,\ b] \longrightarrow \mathbb{R}$ are increasing, then $g, h \in \mathrm{BV}[a,\ b]$ and their difference $f = g - h$ is also in $\mathrm{BV}[a,\ b]$. □

Corollary (1)

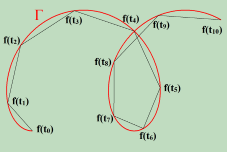
If $f \in \mathrm{BV}[a,\ b]$, then f is differentiable a.e. on $[a,\ b]$ and $f' \in \mathcal{L}(a,\ b)$

Proof.

Let $f \in \mathrm{BV}[a,\ b]$, then there exist increasing functions g, h on $[a,\ b]$ such that $f = g - h$ (Jordan Decomposition Theorem). The increasing functions g and h are differentiable a.e. in $[a,\ b]$ and $g', h' \in \mathcal{L}(a,\ b)$ (see Lecture 20). Therefore f is differentiable a.e. and $f' = g' - h' \in \mathcal{L}(a,\ b)$ □

Rectifiable Curves

Let $\mathbf{f} : [a,\ b] \longrightarrow \mathbb{R}^n$ be a vector valued function $\mathbf{f}(t) = (f_1(t), \cdots, f_n(t))$. As t runs through the interval $[a,\ b]$ $\mathbf{f}(t)$ describes a curve Γ in $\mathbb{R}^n$, the graph of $\mathbf{f}$. The curve Γ is given parametrically by $\mathbf{f}$.
The length of Γ can be thought of as the supremum of the lengths of the inscribed polygonal curves. Let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a,\ b]$ let $Q_j = \mathbf{f}(t_j)$. The polygonal curve connecting Q_0 to Q_1, Q_1 to $Q_2, \cdots, Q_{n-1}$ to Q_n is an approximation of Γ.



The length of the polygonal curve is $\sum_{j=1}^{n} \text{dis}(Q_{j-1}Q_j)$. Define the length $L(\Gamma)$ of Γ as the supremum over the lengths all such polygonal curves:

$$L(\Gamma) = \sup\{S(\mathbf{f}, P) :\ P \text{ partition of } [a,\ b]\}$$

where

$$S(\mathbf{f}, P) = \sum_{j=1}^{n} \|\mathbf{f}(t_j) - \mathbf{f}(t_{j-1})\| = \sum_{j=1}^{n} \left[\sum_{k=1}^{n} (f_k(x_j) - f_k(x_{j-1}))^2\right]^{1/2}$$

The curve Γ is said to be rectifiable if $L(\Gamma) < \infty$.

Theorem (3)

$\mathbf{f} = (f_1, \cdots, f_n) : [a,\ b] \longrightarrow \mathbb{R}^n$ and Γ the graph of $\mathbf{f}$. Then Γ is rectifiable if and only $f_j \in \mathrm{BV}[a,\ b]$ for $j = 1, \cdots, n$. Moreover

$$\max(V(f_1), \cdots, V(f_n)) \ \leq\ L(\Gamma) \ \leq\ \sum_{j=1}^{n} V(f_j)$$

Proof.

Observe that if $c_1, \cdots, c_n \in \mathbb{R}$, then we have the following inequalities

$$\max(|c_1|, \cdots, |c_n|) \ \leq\ \left(\sum_{j=1}^{n} c_j^2\right)^{1/2} \ \leq\ \sum_{j=1}^{n} |c_j|$$

Let $P = \{t_0, \cdots, t_n\}$ be a partition of $[a,\ b]$. It follows from the above observation that

$$\max_{1\leq j\leq n} S(f_j, P) \ \leq\ S(\mathbf{f}, P) \ \leq\ \sum_{j=1}^{n} S(f_j, P)\ .$$

By passing to the suprema, we get

$$\max_{1\leq j\leq n} V(f_j) \ \leq\ L(\Gamma) \ \leq\ \sum_{j=1}^{n} V(f_j)\ .$$

□