Real Analysis MAA 6616
Lecture 22
Absolutely Continuous Functions

A function $f : [a,\ b] \longrightarrow \mathbb{R}$ is absolutely continuous on $[a,\ b]$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any countable (finite or infinite) collection of non overlapping intervals $\{I_j = [a_j,\ b_j]\}_j$ in $[a,\ b]$ we have

$$\sum_j (b_j - a_j) < \delta \implies \sum_j |f(b_j) - f(a_j)| < \epsilon .$$

Denote by $\mathrm{AC}[a,\ b]$ the space of absolutely continuous functions on $[a,\ b]$. Note that if $f \in \mathrm{AC}[a,\ b]$, then f is (uniformly) continuous on $[a,\ b]$. However the converse is not true.

Lemma (1)

The Cantor-Lebesgue function $\phi : [0,\ 1] \longrightarrow [0,\ 1]$ is continuous increasing but it is not absolutely continuous.

Proof.

Recall that the Cantor function ϕ is constant on each (removed) middle third interval in the construction of the Cantor set and that ϕ is increasing and $\phi(0) = 0$, $\phi(1) = 1$.

At the first step in the construction of the Cantor set. We have the two remaining intervals $[a_1,\ b_1] = [0,\ 1/3]$ and $[a_2,\ b_2] = [2/3,\ 1]$ so that

$$\sum_{j=1}^{2} (b_j - a_j) = \frac{2}{3} \quad \text{and} \quad \sum_{j=1}^{2} \phi(b_j) - \phi(a_j) = 1$$

At the second step C_2 is the union of the 2^2 intervals of length 3^{-2}: $[a_1,\ b_1] = [0,\ 1/9]$, $[a_2,\ b_2] = [2/9,\ 3/9]$, $[a_3,\ b_3] = [6/9,\ 7/9]$, and $[a_4,\ b_4] = [8/9,\ 9/9]$. Hence

$$\sum_{j=1}^{2^2} (b_j - a_j) = \left(\frac{2}{3}\right)^2 \quad \text{and} \quad \sum_{j=1}^{2^2} \phi(b_j) - \phi(a_j) = 1$$

In general at the n-th step we get C_n as the disjoint union of 2^n intervals $[a_j,\ b_j]$ each with length 3^{-n} so that

$$\sum_{j=1}^{2^n} (b_j - a_j) = \left(\frac{2}{3}\right)^n \quad \text{and} \quad \sum_{j=1}^{2^n} \phi(b_j) - \phi(a_j) = 1$$

It follows that for $\epsilon = 1/2$, the condition for absolute continuity does not hold since we have a collection of finitely many intervals with total measure $(2/3)^n$ which can be made as small as we which while total variation of ϕ is 1. □

Denote by $\mathrm{Lip}[a,\ b]$ the space of Lipschitz function on $[a,\ b]$. That is $f \in \mathrm{Lip}[a,\ b]$ if and only if there exists $c > 0$ such that $|f(x) - f(y)| \leq c\,|x - y|$ for all $x, y \in [a,\ b]$.

Theorem (1)

A Lipschitz function on $[a,\ b]$ is absolutely continuous on $[a,\ b]$: $\mathrm{Lip}[a,\ b] \subset \mathrm{AC}[a,\ b]$

Proof.

If $[a_j,\ b_j] \subset [a,\ b]$, then $\left|f(b_j) - f(a_j)\right| \leq c\left|b_j - a_j\right|$. Hence

$$\sum_j \left|f(b_j) - f(a_j)\right| \leq c \sum_j \left|b_j - a_j\right|$$

Therefore for $\epsilon > 0$, we can take $\delta = \epsilon/c$ for f to satisfy the definition of absolute continuity. □

There exist absolutely continuous functions that are not Lipschitz continuous as illustrated below.

The Function $f(x) = \sqrt{s}$ is in $\mathrm{AC}[0,\ 1]$ but not in $\mathrm{Lip}[0,\ 1]$. First we verify $\sqrt{x} \notin \mathrm{Lip}[0,\ 1]$. If it were Lipschitz, then there would be $c > 0$ such that for every $0 \leq x < y \leq 1$, we would have $\sqrt{y} - \sqrt{x} \leq c(y - x)$. In particular for $x = 0$ we would have $\sqrt{y} \leq cy$ for all $y \in (0,\ 1)$. This means $1 \leq c\sqrt{y}$ for all $y > 0$ which is absurd.

However $f \in \mathrm{Lip}[\alpha,\ 1]$, with Lipschitz constant $c = \dfrac{1}{2\sqrt{\alpha}}$ if $\alpha > 0$. Indeed for $\alpha \leq x < y \leq 1$, we have

$$\sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}} \leq \frac{1}{2\sqrt{\alpha}}(y - x).$$

Now we prove that $\sqrt{x} \in \mathrm{AC}[0,\ 1]$. Given $\epsilon > 0$, let $\delta = \dfrac{\epsilon^2}{2}$. Let $\{I_k = [u_k,\ v_k]\}_k$ be a countable collection of non overlapping intervals in $[0,\ 1]$ such that $\sum_k \ell(I_k) < \delta$. Consider the point $x_0 = \epsilon^2/4$, there exists at most one interval I_{k_0} that contains x_0 in its interior. In which we split I_{k_0} into two intervals $[u_{k_0},\ x_0]$ and $[x_0,\ v_{k_0}]$. Let C_1 be the collection of intervals I_k contained in $[0,\ x_0]$ and C_2 be the collection of intervals I_k contained in $[x_0,\ 1]$. Using the fact that $\sqrt{x}$ is an increasing function, we have $\displaystyle\sum_{k,\ I_k \in C_1} (\sqrt{v_k} - \sqrt{u_k}) \leq \sqrt{x_0} = \frac{\epsilon}{2}$. Using the fact that $\sqrt{x} \in \mathrm{Lip}[x_0,\ 1]$ with Lipschitz constant $c = 1/2\sqrt{x_0} = 1/\epsilon$, we have $\displaystyle\sum_{k,\ I_k \in C_2} (\sqrt{v_k} - \sqrt{u_k}) \leq \frac{1}{\epsilon} \sum_{k,\ I_k \in C_2} \ell(I_k) \leq \frac{\epsilon}{2}$. This shows that $\sqrt{x} \in \mathrm{AC}[0,\ 1]$

Theorem (2)

An absolutely continuous function on $[a,\ b]$ *is of bounded variation on* $[a,\ b]$*:* $\mathrm{AC}[a,\ b] \subset \mathrm{BV}[a,\ b]$. *Moreover, an absolutely continuous function can be written as the difference of two increasing absolutely continuous functions*

Proof.

Let $f \in \mathrm{AC}[a,\ b]$, we need to show $f \in \mathrm{BV}[a,\ b]$. Let $\epsilon = 1$ and $\delta > 0$ be a corresponding positive number so that f satisfies the absolute continuity property for the pair (ϵ, δ). Let $N \in \mathbb{N}$ be such that $N > \dfrac{b-a}{\delta}$ and for $i = 0, \cdots, N$ let $x_i = a + i\frac{b-a}{N}$ so $P = \{x_i\}_{i=0}^N$ be a partition of $[a,\ b]$ by equally spaces points and $x_{i+1} - x_i < \delta$. Now let Q be any partition of $[a,\ b]$ and for every $i = 1, \cdots, N$, let $Q_i = (Q \cap [x_{i-1},\ x_i]) \cup \{x_{i-1}, x_i\}$ so that Q_i is a partition of the interval $[x_{i-1},\ x_i]$. Since $x_i - x_{i-1} < \delta$, then $V(f_{[x_{i-1},\ x_i]}, Q_i) < 1$ (absolute continuity of f restricted to $[x_{i-1},\ x_i]$). Hence,

$$V(f, Q) \leq \sum_{i=1}^{N} V(f_{[x_{i-1},\ x_i]}, Q_i) \leq \sum_{i=1}^{N} 1 = N.$$

Therefore $V(f, [a,\ b]) \leq N$ and $f \in \mathrm{BV}[a,\ b]$.

Now we prove that f can be written as the difference of two absolutely continuous and increasing functions. As was done earlier (Lecture 21), since $f \in \mathrm{BV}[a,\ b]$ we can write f as the difference of two increasing functions. Namely, $f = g - h$, where $h(x) = V(f, [a,\ x])$ and $g(x) = f(x) + V(f, [a,\ x])$. To complete the proof, we need to verify that h is absolutely continuous. Let $\epsilon >$ and let $\delta > 0$ so that f satisfies the absolute continuity property for the pair $\epsilon/2,\ \delta$. Let $\{I_k = [u_k,\ v_k]\}_{k=1}^n$ be a collection of disjoint subintervals of $[a,\ b]$ such that $\sum_k \ell(I_k) < \delta$. For $k \in \{1, \cdots, n\}$, let P_k be a partition of the interval I_k. Then $\displaystyle\sum_{k=1}^{n} V(f_{I_k}, P_k) < \frac{\epsilon}{2}$. By taking the supremum over each partition P_k of I_k, we get $\displaystyle\sum_{k=1}^{n} V(f_{I_k}) \leq \epsilon/2$. Since $V(f_{I_k}) = h(v_k) - h(u_k)$, we have proved

$$\sum_{k=1}^{n} (v_k - u_k) < \delta \implies \sum_{k=1}^{n} (h(v_k) - h(u_k)) < \epsilon.$$

That is $h \in \mathrm{AC}[a,\ b]$.

Let $f \in \mathcal{L}(a, b)$. Extend f to the interval $[a, b+1]$ by defining it as $f(x) = f(b)$ for any $x \in (b, b+1]$. For any $0 < h < 1$ define the divided difference function $\mathrm{D}_h f$ and the average function $\mathrm{Av}_h f$ on $[a, b]$ by

$$\mathrm{D}_h f(x) = \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \mathrm{Av}_h f(x) = \frac{1}{h} \int_x^{x+h} f(s) ds \, .$$

Note that if $[c, d] \subset [a, b]$, then $\displaystyle\int_c^d \mathrm{D}_h f(x) dx = \mathrm{Av}_h f(d) - \mathrm{Av}_h f(c)$.

Recall that a collection $\mathcal{C}$ of measurable functions on a set E is said to be uniformly integrable over E if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $f \in \mathcal{C}$ we have $\displaystyle\int_A |f| \, dx < \epsilon$ for all $A \subset E$ with $m(A) < \delta$.

Theorem (3)

Let f be a continuous function on a closed and bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\mathrm{D}_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

Proof.

"$\Longleftarrow$" Suppose that $\{\mathrm{D}_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$. Let $\epsilon > 0$ and a corresponding $\delta > 0$ such that for every $h \in (0, 1]$ we have $\displaystyle\int_A |\mathrm{D}_h f| \, dx < \epsilon$ whenever $A \subset [a, b]$ has measure $m(A) < \delta$.

Now let $\{I_k = (u_k, v_k)\}_{k=1}^n$ be a collection of disjoint intervals in $[a, b]$ such that $\sum_k \ell(I_k) < \delta$. Let $E = \bigcup_k I_k$. We have $m(E) < \delta$. For any $h \in (0, 1]$ and any index $k = 1, \cdots, n$ we have $\mathrm{Av}_h f(v_k) - \mathrm{Av}_h f(u_k) = \displaystyle\int_{u_k}^{v_k} \mathrm{D}_h f(x) dx$

Therefore for every $h \in (0, 1]$ we

$$\sum_{k=1}^n |\mathrm{Av}_h f(v_k) - \mathrm{Av}_h f(u_k)| \leq \sum_{k=1}^n \int_{I_k} |\mathrm{D}_h f(x)| \, dx = \int_E |\mathrm{D}_h f(x)| \, dx < \epsilon \, .$$

Since f is continuous, then $\mathrm{Av}_h f(x) \longrightarrow f(x)$ as $h \longrightarrow 0$, then it follows from the passage to the limit in the above inequality that $\displaystyle\sum_{k=1}^n |f(v_k) - f(u_k)| \leq \epsilon$ and $f \in \mathrm{AC}[a, b]$.

Proof.

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"$\Longrightarrow$" Suppose $f \in \mathrm{AC}[a,b]$. Then we can express f as $f = g - h$ with $g, h \in \mathrm{AC}[a,b]$ and increasing. We can therefore assume that f is increasing. This implies that the divided difference functions $\mathrm{D}_h f$ are nonnegative. Observe that if $[\alpha,\ \beta] \subset [a,\ b+1]$ then, using change of variables, we have

$$\int_\alpha^\beta \mathrm{D}_h f(x)dx = \frac{1}{h}\left[\int_{\alpha+h}^{\beta+h} f(s)ds - \int_\alpha^\beta f(s)ds\right] = \frac{1}{h}\int_0^h m(t)dt$$

where $m(t) = f(\beta+t) - f(\alpha+t)$.

Now we prove that $\{\mathrm{D}_h f\}_{0<h\le 1}$ is uniformly integrable over $[a,\ b]$. Let $\epsilon > 0$ and let $\delta > 0$ such that f satisfies the absolute continuity property for the pair ϵ', δ with $\epsilon' < \epsilon$. Thus if $\{I_k = (u_k,\ v_k)\}_{k=1}^n$ is a collection of disjoint intervals in $[a,\ b]$ such that $\sum_k \ell(I_k) < \delta$, then $\sum_k (f(v_k) - f(u_k)) < \epsilon'$. Note that for any $0 \le t \le 1$, we have $\sum_k \ell(t + I_k) < \delta$, and $\sum_k (f(v_k+t) - f(u_k+t)) < \epsilon'$. Let $U = \bigcup_k I_k$. Then $m(U) < \delta$ and it follows from the above observations that

$$\int_U \mathrm{D}_h f(x)dx = \frac{1}{h}\int_0^h \sum_{k=1}^n (f(v_k+t) - f(u_k+t))dt < \epsilon'$$

Let $E \subset [a,\ b]$ be such that $m(E) < \delta/2$. There exists a G_δ set G such that $E \subset G$ and $m(G) = m(E)$. The set G can be written as $G = \bigcap_n U_n$ where $\{U_n\}_n$ is nested collection of open set. Then there exists $p \in \mathbb{N}$ such that $E \subset U_p$ and $m(U_p) < 2\delta/3$. Now $U_p = \bigcup_k V_{k,p}$ where $V_{k,p}$ is a disjoint union of a finite collection of open intervals and $V_{k,p} \subset V_{k+1,p}$. Since $m(V_{k,p}) < m(U_p) < 2\delta/3$, then it follows from the above estimate that $\int_{V_{k,p}} \mathrm{D}_h f(x)dx < \epsilon'$ for all k. Hence

$$\int_{U_p} \mathrm{D}_h f(x)dx = \lim_{k\to\infty} \int_{V_{k,p}} \mathrm{D}_h f(x)dx \le \epsilon'.$$

Finally

$$\int_E \mathrm{D}_h f(x)dx \le \int_{U_p} \mathrm{D}_h f(x)dx = \lim_{k\to\infty} \int_{V_{k,p}} \mathrm{D}_h f(x)dx \le \epsilon' < \epsilon.$$

$\square$

Theorem (4)

Let $f \in \mathrm{AC}[a,\ b]$. Then f is differentiable a.e. on $[a,\ b]$, $f' \in \mathcal{L}(a,\ b)$, and

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof.

Since $f \in \mathrm{AC}[a,\ b]$, then it follows from

$$\int_a^b \mathrm{D}_h f(x)dx = \mathrm{Av}_h f(b) - \mathrm{Av}_h f(a) = \frac{1}{h}\int_b^{b+h} f(x)dx - \int_a^{a+h} f(x)dx\,,$$

by letting $h \to 0$ that

$$\lim_{h\to 0} \frac{1}{h}\left[\int_a^b (f(x+h) - f(x))dx\right] = f(b) - f(a).$$

The function f can be written as $f = g - h$ with $g, h \in \mathrm{AC}[a,\ b]$ increasing. Then f is differentiable a.e. and $f' \in \mathcal{L}(a,\ b)$ (Corollary 1 in Lecture 21). Therefore $\mathrm{D}_{1/n} f \longrightarrow f'$ pointwise a.e. in $[a,\ b]$. We know from Theorem 3 that the collection $\{\mathrm{D}_{1/n} f\}_n$ is uniformly integrable. Consequently the Vitali Convergence Theorem (Lecture 16) implies

$$f(b) - f(a) = \lim_{n\to\infty} \int_a^b \mathrm{D}_{1/n} f(x)dx = \int_a^b \lim_{n\to\infty} \mathrm{D}_{1/n} f(x)dx = \int_a^b f'(x)dx$$

□

Let $g \in \mathcal{L}(a,\ b)$, the function $f : [a,\ b] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \lambda + \int_a^x g(t)dt\,,$$

$\lambda \in \mathbb{R}$ constant, is called an indefinite integral of g over $[a,\ b]$.

Theorem (5)

We have the following: $f \in \mathrm{AC}[a,\ b]$ if and only if f is an indefinite integral (of f').

Proof.

"$\Longrightarrow$" Let $f \in \mathrm{AC}[a,\ b]$, then f differentiable a.e. and $f' \in \mathcal{L}(a,\ b)$. For any $x \in [a,\ b]$ we have $f \in \mathrm{AC}[a, x]$ and Theorem 4 gives $f(x) = f(a) + \int_a^x f'(t)dt$. Therefore f is an indefinite integral of f'.

"$\Longleftarrow$" Suppose that f is an indefinite integral of $g \in \mathcal{L}(a, b)$: $f(x) = f(a) + \int_a^x g(t)dt$.

Let $\epsilon > 0$. It follows from $|g| \in \mathcal{L}(a,\ b)$ that there exists $\delta > 0$ such that $\int_E |g|dx < \epsilon$ whenever $E \subset [a,\ b]$ satisfies $m(E) < \delta$ (Proposition 1, Lecture 17). Let $\{I_k = (u_k,\ v_k)\}_{k=1}^n$ be a collection of disjoint open intervals in $[a,\ b]$ such that $\sum_k \ell(I_k) < \delta$. Let $E = \bigcup_k I_k$. Then

$$\sum_{k=1}^n |f(v_k) - f(u_k)| = \sum_{k=1}^n \left| \int_{u_k}^{v_k} g(t)dt \right| \leq \sum_{k=1}^n \int_{u_k}^{v_k} |g(t)|\, dt = \int_E |g(t)|dt < \epsilon$$

Therefore $f \in \mathrm{AC}[a,\ b]$. □

Corollary (1)

Let $f : [a,\ b] \longrightarrow \mathbb{R}$ be a monotone function. Then $f \in \mathrm{AC}[a, b]$ if and only if $\int_a^b f'(x)dx = f(b) - f(a)$.

Proof.

"$\Longrightarrow$" This is a consequence of Theorem 5.

"$\Longleftarrow$" Suppose that f is monotone (increasing). We know from Lebesgue's Theorem that f is differentiable a.e. moreover $\int_c^d f'(x)dx \leq f(d) - f(c)$ for all $a \leq c \leq d \leq b$. Suppose further that $\int_a^b f'(x)dx = f(b) - f(a)$. Let $x \in [a,\ b]$. Then

$$0 = \int_a^b f'(t)dt - (f(b) - f(a)) = \left[\int_a^x f'(t)dt - (f(x) - f(a))\right] + \left[\int_x^b f'(t)dt - (f(b) - f(x))\right] \leq 0$$

This means $f(x) = f(a) + \int_a^x f'(t)dt$ (f an indefinite integral) and consequently $f \in \mathrm{AC}[a,\ b]$ by Theorem 5. □

Lemma (2)

Let $f \in \mathcal{L}(a,\ b)$*. Then* $f = 0$ *a.e. on* $[a,\ b]$ *if and only if* $\int_x^y f(t)dt = 0$ *for all* $a \le x \le y \le b$*.*

Proof.

"$\Longleftarrow$" Suppose that $f \in \mathcal{L}(a,\ b)$ and $\int_x^y f(t)dt = 0$ for all $a \le x \le y \le b$. If $U \subset [a,\ b]$ is open, then $U = \bigcup_k I_k$, where $\{I_k = (u_k,\ v_k)\}_k$ is a countable collection of disjoint intervals in $[a,\ b]$, then it follows from the additive property of the integral that $\int_U f(x)dx = 0$. Now if $G \subset [a,\ b]$ is a G_δ set, then we can write $G = \bigcap_n U_n$ for some collection $\{U_n\}_n$ of nested open sets in $[a,\ b]$. Then $\int_G fdx = \lim_{n\to\infty} \int_{U_n} fdx = 0$. Next, if $E \subset [a,\ b]$ is an arbitrary measurable set, then there exists a G_δ set G such that $E \subset G$ and $m(G \backslash E) = 0$. Hence

$$\int_E f(x)dx = \int_G f(x)dx - \int_{G\backslash E} f(x)dx = 0.$$

Let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ be the positive and negative parts of f. Both are nonnegative integrable functions and $f = f^+ - f^-$. Let

$$E^+ = (f^+)^{-1}(\mathbb{R}) = \{x \in [a,\ b] : f(x) \ge 0\} \text{ and } E^- = (f^-)^{-1}(\mathbb{R}) = \{x \in [a,\ b] : f(x) \le 0\}.$$

Then $\int_a^b f^\pm(x)dx = \pm \int_{E^\pm} f(x)dx = 0$. We know that if a nonnegative function has a vanishing integral, then the function is 0 a.e. Thus $f^\pm = 0$ a.e. and consequently $f = f^+ - f^- = 0$ a.e. on $[a,\ b]$. □

Theorem (6)

Let $f \in \mathcal{L}(a,\ b)$*. Then* $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ *for almost all* $x \in [a,\ b]$*.*

Proof.

Define the function F on $[a,\ b]$ by $F(x) = \int_a^x f dt$. Then as an indefinite integral $F \in \mathrm{AC}[a,\ b]$, F is differentiable and $F' \in \mathcal{L}([a,\ b])$ (Theorem 5). Now we need to verify that $F' - f = 0$ a.e. on $[a,\ b]$. For this it is enough to verify that if $a \le x_1 < x_2 \le b$, then $\int_{x_1}^{x_2} (F' - f) dt = 0$ (Lemma 2):

$$\begin{aligned} \int_{x_1}^{x_2} (F' - f) dt &= \int_{x_1}^{x_2} F' dt - \int_{x_1}^{x_2} f dt = F(x_2) - F(x_1) - \int_{x_1}^{x_2} f dt \\ &= \int_a^{x_2} f dt - \int_a^{x_1} f dt - \int_{x_1}^{x_2} f dt = 0. \end{aligned}$$

□

A function $s \in \mathrm{BV}[a,\ b]$ is said to be singular if $s' = 0$ a.e. on $[a,\ b]$. The Cantor-Lebesgue function ϕ is an example of a nonconstant singular function. It follows from Theorem 4 that if an absolutely continuous is singular, then it is constant. The following theorem (Lebesgue) gives a decomposition of a function with bounded variation as the sum of an absolutely continuous function and a singular function.

Theorem (7)

Let $f \in \mathrm{BV}[a,\ b]$. Then f can be written as $f = g + h$, with $g \in \mathrm{AC}[a,\ b]$ and h a singular function.

Proof.

Since $f \in \mathrm{BV}[a,\ b]$, then f is differentiable a.e. and $f' \in \mathcal{L}(a,\ b)$. Let $g = \int_a^x f' dt$ and $h = g - f$. Then $g \in \mathrm{AC}[a,\ b]$, $h \in \mathrm{BV}[a,\ b]$ and $g' = f'$ and $h' = 0$

□