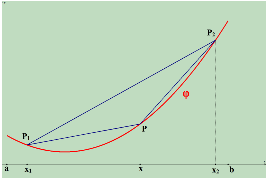
Real Analysis MAA 6616
Lecture 23
Convex Functions

A function $\phi : (a,\ b) \longrightarrow \mathbb{R}$ is said to be convex in $(a,\ b)$ if for every interval $[x_1,\ x_2] \subset (a,\ b)$, the graph of ϕ over $[x_1,\ x_2]$ lies below or on the the segment line joining the points $P_1 = (x_1, \phi(x_1))$ and $P_2 = (x_2, \phi(x_2))$.



Since the segment P_1P_2 can be parameterized as

$$x = tx_1 + (1-t)x_2, \quad y = t\phi(x_1) + (1-t)\phi(x_2) \ \text{ with } \ 0 \le t \le 1,$$

then ϕ is convex if and only if for every $a < x_1 \le x_2 < b$ and for every $0 \le t \le 1$ we have

$$\phi(tx_1 + (1-t)x_2) \le t\phi(x_1) + (1-t)\phi(x_2).$$

An equivalent formulation is the following: ϕ is convex if and only if for every $s_1 \ge 0$, $s_2 \ge 0$ such that $s_1 + s_2 > 0$ and for every $a < x_1 \le x_2 < b$ we have

$$\phi\left(\frac{s_1}{s_1+s_2}x_1 + \frac{s_2}{s_1+s_2}x_2\right) \le \frac{s_1\phi(x_1) + s_2\phi(x_2)}{s_1+s_2}.$$

By repeated use of this characterization we get the discrete Jensen's Inequality

Proposition (1)

Let $\phi : (a,\ b) \longrightarrow \mathbb{R}$ be a convex function. Then for every $\{x_j\}_{j=1}^N \subset (a,\ b)$ and for every $\{s_j\}_{j=1}^N \subset \mathbb{R}$ with $s_j \ge 0$ and $\sum_j s_j > 0$, we have

$$\phi\left(\frac{\sum_{j=1}^N s_j x_j}{\sum_{j=1}^N s_j}\right) \le \frac{\sum_{j=1}^N s_j \phi(x_j)}{\sum_{j=1}^N s_j}$$

Lemma (1)

A function $\phi : (a,\ b) \longrightarrow \mathbb{R}$ is convex if and only if for every $a < x_1 < x < x_2 < b$, we have

$$\frac{\phi(x) - \phi(x_1)}{x - x_1} \leq \frac{\phi(x) - \phi(x_2)}{x - x_2} \iff \text{Slope}(PP_1) \leq \text{Slope}(PP_2)$$

where P_1, P, and P_2 are the points on the graph of ϕ corresponding to x_1, x, and x_2

Proof.

Let $a < x_1 < x < x_2 < b$, then $x = tx_1 + (1-t)x_2$ with $t = \dfrac{x_2 - x}{x_2 - x_1}$ and $1 - t = \dfrac{x - x_1}{x_2 - x_1}$. Hence if ϕ is convex then

$$\phi(x) \leq \frac{x_2 - x}{x_2 - x_1}\phi(x_1) + \frac{x - x_1}{x_2 - x_1}\phi(x_2)\ .$$

The Lemma follows from this inequality. □

Proposition (2)

Let $\phi : (a,\ b) \longrightarrow \mathbb{R}$. If ϕ is differentiable and ϕ' is increasing in $(a,\ b)$, then ϕ is convex. In particular, if ϕ has a nonnegative second derivative then ϕ is convex

Proof.

We need to show that ϕ satisfies the condition in Lemma 1. Let $a < x_1 < x < x_2 < b$. Then from the Mean Value Theorem and ϕ' increasing, there exists $c_1 \in (x_1,\ x)$ and $c_2 \in (x,\ x_2)$ such that

$$\left[\frac{\phi(x) - \phi(x_1)}{x - x_1} = \phi'(c_1)\right] \leq \left[\phi'(c_2) = \frac{\phi(x) - \phi(x_2)}{x - x_2}\right]$$

□

As a consequence, we have the following

- If $p \geq 1$, then x^p is convex on $(0,\ \infty)$.
- If $\lambda \in \mathbb{R}$, then $\mathrm{e}^{\lambda x}$ is convex on $(-\infty,\ \infty)$.
- $\log \dfrac{1}{x} = -\log x$ is convex on $(0,\ \infty)$.

Convex functions satisfy the following properties

- If ϕ and ψ are convex on $(a,\ b)$, then so is $\phi+\psi$.
- If ϕ is convex on $(a,\ b)$ and $c>0$, then $c\phi$ is also convex on $(a,\ b)$
- If $\{\phi_n\}_n$ is a sequence of convex functions on $(a,\ b)$ and if $\phi_n \longrightarrow \phi$ pointwise in $(a,\ b)$, then the limit ϕ is also convex.

For a function $f:(a,\ b)\longrightarrow \mathbb{R}$ and $c\in(a,\ b)$, the right-hand derivative $f'(c^+)$ and left-hand derivative $f'(c^-)$ are:

$$f'(c^+)=\lim_{h\to 0+}\frac{f(c+h)-f(c)}{h} \quad \text{and} \quad f'(c^-)=\lim_{h\to 0-}\frac{f(c+h)-f(c)}{h}$$

(provided the limits exist and are finite).

Proposition (3)

Let $\phi:(a,\ b)\longrightarrow\mathbb{R}$ be convex. Then

1. *$\phi'(x^\pm)$ exist for all $x\in(a,\ b)$. Moreover if $a<x_1<x<x_2<b$, then*
$$\phi'(x_1^-)\ \le\ \phi'(x_1^+)\ \le\ \frac{\phi(x_2)-\phi(x_1)}{x_2-x_1}\ \le\ \phi'(x_2^-)\ \le\ \phi'(x_2^+)$$
2. *For every closed interval $[c,\ d]\subset(a,\ b)$, ϕ is Lipschitz on $[c,\ d]$ and consequently $\phi\in \mathrm{AC}[c,\ d]$.*

Proof.

Part 1 is a direct consequence of Lemma1.

To verify part 2, let $[c,\ d]\subset(a,\ b)$ and let $M=\max\{\left|\phi'(c^+)\right|,\ \left|\phi'(d^-)\right|\}$. It follows from part 1 that for every $x,y\in[c,\ d]$ we have $|\phi(y)-\phi(x)|\ \le\ M\,|y-x|$. □

Theorem (1)

Let $\phi : (a,\ b) \longrightarrow \mathbb{R}$ be convex. Then $\phi'(x)$ exists except at most on a countable set. Furthermore ϕ' is increasing.

Proof.

Consider the functions ϕ'_+ and ϕ'_- defined on $(a,\ b)$ by $\phi'_\pm(x) = \phi'(x^\pm)$. It follows from Proposition 3 that $\phi'_\pm$ are increasing functions and $\phi'_- \le \phi'_+$. Since an increasing function is continuous except possible at a countable set, then there exists a countable set $C \subset (a,\ b)$ such that $\phi'_\pm$ are continuous on $E = (a,\ b)\backslash C$. Let $x_0 \in E$ and let $\{x_n\}_n$ be a decreasing sequence that converges to x_0. It follows from

$$\phi'_-(x_0) \le \phi'_+(x_0) \le \frac{\phi(x_n) - \phi(x_0)}{x_n - x_0} \le \phi'_-(x_n) \le \phi'_+(x_n)$$

that by letting $n \to \infty$, that we have $\phi'_-(x_0) \le \phi'_+(x_0) \le \phi'_-(x_0)$. Hence $\phi'_-(x_0) = \phi'_+(x_0)$ and so ϕ is differentiable at x_0. □

A line L through a point $P = (x_0, \phi(x_0))$ on the graph of a convex function ϕ is said to be a supporting line for ϕ if L lies below the graph of ϕ.

Lemma (2)

Let L be a line with equation $y = \alpha(x - x_0) + \phi(x_0)$. Then L is a supporting line for the convex function ϕ if and only if $\phi'(x_0^-) \le \alpha \le \phi'(x_0^+)$

Proof.

Suppose that L is a supporting line for ϕ through the point $(x_0, \phi(x_0))$. Let $x \in (a,\ x_0)$. Then $\phi(x) - \alpha(x - x_0) + \phi(x_0) \ge 0$ it follows that $\dfrac{\phi(x) - \phi(x_0)}{x - x_0} \le \alpha$. By letting $x \to x_0^-$ we get $\phi'(x_0^-) \le \alpha$. A similar argument gives $\alpha \le \phi'(x_0^+)$. The proof of the converse is left as an exercise. □

Theorem (2)

Let $E \subset \mathbb{R}^q$ be a measurable set with $0 < m(E) < \infty$. Let $f \in \mathcal{L}(E)$ and f finite a.e. on E and let ϕ be a convex function on $\mathbb{R}$. Then

$$\phi\left(\frac{1}{m(E)} \int_E f dx\right) \leq \frac{1}{m(E)} \int_E (\phi \circ f) dx$$

Proof.

Since f is finite a.e., there exists a set $Z \subset E$ with $m(Z) = 0$ and a $M > 0$ such that $f(x) \in (-M,\ M)$ for every $x \in E \backslash Z$. Let $c = \dfrac{1}{m(E)} \displaystyle\int_E f dx$. Then $c \in (-M,\ M)$. Let $\alpha \in \mathbb{R}$ such that $\phi'(c^-) \leq \alpha \leq \phi'(c^+)$ so that the line with equation $y = \alpha(t - c) + \phi(c)$ is a supporting line through $(c,\ \phi(c))$ for the function ϕ. Hence for every $x \in E \backslash Z$ we have

$$\phi(f(x)) \geq \alpha\,(f(x) - c) + \phi(c).$$

Now integrate this inequality over E to get

$$\int_E \phi(f(x)) dx \geq \alpha\left(\int_E f(x) dx - cm(E)\right) + \phi(c)\, m(E) = \phi(c)\, m(E).$$

The conclusion follows by rearranging this inequality. □