Real Analysis MAA 6616
Lecture 24
L^p Spaces: Norm

Let $E \subset \mathbb{R}^n$ be a measurable set and consider the space of measurable functions on E that are finite a.e.:

$$\mathcal{M}_0(E) = \left\{f : E \longrightarrow \overline{\mathbb{R}} : \, f \text{ measurable and finite a.e. on } onE\right\}.$$

Consider the relation $\cong$ on $\mathcal{M}_0(E)$ given by $f \cong g \iff f = g$ a.e. on E. Then $\cong$ is an equivalence relation (exercise). Let $\mathcal{M}_0(E)/_{\cong}$ be the quotient space. Hence $\mathcal{M}_0(E)/_{\cong}$ is the set of all equivalence classes $\mathbf{x} = [f] = \{h \in \mathcal{M}_0(E) : f \cong h\}$. It can be verified that $\mathcal{M}_0(E)/_{\cong}$ is a vector space over $\mathbb{R}$. For instance, if $\mathbf{x} = [f]$ and $\mathbf{y} = [g]$ are in $\mathcal{M}_0(E)/_{\cong}$ and if $a, b \in \mathbb{R}$, then $a\mathbf{x} + b\mathbf{y} = \mathbf{z} = [af + bg]$ is well defined. Indeed, if $f \cong f'$ and $g \cong g'$, then $(af + bg) \cong (af' + bg')$.

When there is no ambiguity, we will identify $\mathbf{x} = [f] \in \mathcal{M}_0(E)/_{\cong}$ with its representative $f \in \mathcal{M}_0(E)$ or with any other $h \in \mathcal{M}_0(E)$ such that $h \cong f$. For example if we say that $\mathbf{x} = [f] \in \mathcal{M}_0(E)/_{\cong}$ is continuous, it means that the there exists a continuous function $h : E \longrightarrow \mathbb{R}$ such that $f \cong h$.

Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$. Note that if $f \cong g$, then $|f|^p \in \mathcal{L}(E)$ if and only if $|g|^p \in \mathcal{L}(E)$. Denote by $L^p(E)$ the space of functions (more precisely the space of equivalence classes) in $\mathcal{M}_0(E)/_{\cong}$ that are *p*-integrable over E i.e. $\displaystyle\int_E |f|^p \, dx < \infty$.

$$L^p(E) = \left\{[f] \in \mathcal{M}_0(E)/_{\cong} : \int_E |f|^p \, dx < \infty\right\}.$$

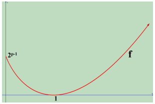
A function $f \in \mathcal{M}_0(E)$ is said to be essentially bounded if there exists $M > 0$ such that $|f(x)| \leq M$ for almost all $x \in E$. Again if $f \cong g$, and f is essentially bounded then so is g. Define $L^\infty(E)$ as the subspace of essentially bounded functions in $\mathcal{M}_0(E)/_{\cong}$.

Lemma (1)

Let $a,\ b\ \geq 0$ *and* $1 \leq p < \infty$. *Then* $(a+b)^p \leq 2^{p-1}(a^p + b^p)$.

Proof.

The inequality is trivial if either $a = 0$ or if $p = 1$. So assume that $a > 0$ and $p > 1$ and let $b = ta$ with $t \geq 0$. The inequality of the lemma is equivalent to $(1+t)^p \leq 2^{p-1}(1+t^p)$ for all $t \geq 0$. Consider the function $f(t) = 2^{p-1}(1+t^p) - (1+t)^p$.



We have $f(0) = 2^{p-1} - 1 > 0$, $f(1) = 0$, and $f(t) \to \infty$ as $t \to \infty$. For $t > 0$, we have $f'(t) = p(2^{p-1}t^{p-1} - (1+t)^{p-1})$. The equation $f'(t) = 0$ has a unique solution at $t = 1$. Therefore $f(1) = 0$ is a global minimum of f and the lemma follows. □

This lemma implies that for $1 \leq p \leq \infty$, $L^p(E)$ is a linear space. For instance, for $1 \leq p < \infty$, and $f, g \in L^p(E)$, then the inequality $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ implies that

$$\int_E |f+g|^p\,dx \leq 2^{p-1}\int_E |f|^p\,dx + 2^{p-1}\int_E |g|^p\,dx < \infty.$$

For $p = \infty$, there exist $Z_f, Z_g \subset E$ with measure 0 and $M_f, M_g > 0$ such that $|f| < M_f$ on $E \backslash Z_f$ and $|g| < M_g$ on $E \backslash Z_g$. Therefore $|f+g| < M_f + M_g$ on $E \backslash (Z_f \cup Z_g)$ and so $f + g \in L^\infty(E)$.

Let X be a vector space over $\mathbb{R}$ (or over $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow [0, \infty)$ is a norm on X if

- $\|\alpha f\| = |\alpha| \; \|f\|$ for all $f \in X$ and $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$)
- $\|f\| = 0$ if and only if $f = 0$
- $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$.

$(X, \|\cdot\|)$ is called a normed space.

Examples

1. For $f \in L^1(E)$, let $\|f\|_1 = \int_E |f| \, dx$. Then $\|\cdot\|_1$ is a norm on $L^1(E)$. Indeed, if $\|f\|_1 = 0$, then $|f| = 0$ a.e. on E and so $f = 0$ in $L^1(E)$ (more precisely $f \cong 0$ and so $[f] = [0]$). Now we verify the triangle inequality. Let $f, g \in L^1(E)$, then f and g are finite a.e. and so is $f + g$. Furthermore $|f + g| \leq |f| + |g|$ a.e. We have therefore
$$\|f + g\|_1 = \int_E |f + g| \, dx \leq \int_E (|f| + |g|) \, dx \leq \int_E |f| \, dx + \int_E |g| \, dx = \|f\|_1 + \|g\|_1$$
Similarly, one can verify $\|\alpha f\|_1 = |\alpha| \; \|f\|_1$.
2. Let $B(0, R)$ be the closed ball with center 0 and raduis $R > 0$ in $\mathbb{R}^n$: $B(0, R) = \{x \in \mathbb{R}^n : \; |x| \leq R\}$. Let $C^0(B(0, R))$ be the space continuous function on $B(0, R)$. For $f \in C^0(B(0, R))$, let $\|f\|_{\max} = \max_{x \in B(0,R)} |f(x)|$.
Then $\left(C^0(B(0, R)), \|\cdot\|_{\max}\right)$ is a normed space. The verification is left as an exercise.

$L^\infty(E)$ is the space of essentially bounded functions: $f \in L^\infty(E)$ (or more precisely $[f]$ for the equivalence relation $\cong$) if there exists a positive constant M such that $|f(x)| \leq M$ for almost all $x \in E$. Define $\|\cdot\|_\infty$ on $L^\infty(E)$ by

$$\|f\|_\infty = \inf\{M : \ |f| \leq M \ \text{ a.e. on } E\}.$$

Now we verify that $\|\cdot\|_\infty$ is a norm.

We first prove that $\|f\|_\infty$ is an essential upper bound of f, i.e. $|f| \leq \|f\|_\infty$ a.e. on E. To see why, let $n \in \mathbb{N}$ and $M_n = \|f\|_\infty + \frac{1}{n}$, then M_n is an upper bound of f and so there exists a set $Z_n \subset E$ with $m(Z_n) = 0$ and such that $|f(x)| \leq M_n$ for every $x \in E \backslash Z_n$. Let $Z = \bigcup_{n=1}^\infty Z_n$. Then $m(Z) = 0$ and for every $x \in E\backslash Z$, we have $|f(x)| \leq M_n$ for all n. Consequently $f(x) \leq \|f\|_\infty$ on $E\backslash Z$. It follows that if $\|f\|_\infty = 0$, then $f = 0$ a.e. on E.

Let $\alpha \in \mathbb{R}^*$. Then

$$\begin{aligned}\|\alpha f\|_\infty &= \inf\{\hat{M} : \ |\alpha f| < \hat{M} \text{ a.e.}\} = \inf\left\{\hat{M} : \ |f| < \frac{\hat{M}}{|\alpha|} \text{ a.e.}\right\} \\ &= \inf\{|\alpha| M : \ |f| < M \text{ a.e.}\} = |\alpha| \ \inf\{M : \ |f| < M \text{ a.e.}\} = |\alpha| \ \|f\|_\infty\end{aligned}$$

If $f, g \in L^\infty(E)$, then there exist $Z_f \subset E$ and $Z_g \subset E$ with measure 0 such that $|f| \leq \|f\|_\infty$ on $E\backslash Z_f$ and $|g| \leq \|g\|_\infty$ on $E\backslash Z_g$. Then for $x \in E\backslash(Z_f \cup Z_g)$ we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

This means $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Let $E \subset \mathbb{R}^n$ be measurable. For $1 < p < \infty$ and $f \in L^p(E)$ defines $\|f\|_p$ by

$$\|f\|_p = \left(\int_E |f|^p \, dx\right)^{\frac{1}{p}} .$$

We will show that $\|\cdot\|_p$ is norm on $L^p(E)$. The first two conditions for a norm as easy to verify: If $\alpha \in \mathbb{R}$, then $\|\alpha f\|_p = |\alpha| \; \|f\|_p$ follows from the linearity of the integral; and if $\|f\|_p = 0$, then $\int_E |f|^p \, dx = 0$ and so $f = 0$ a.e. on E. To verify the triangle inequality, we need to use Young's and Hölder inequalities.

Define the conjugate of a number $1 < p < \infty$ as the number $q = \dfrac{p}{p-1}$ so that $1 < q < \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1$$

The conjugate of 1 is ∞ and the conjugate of ∞ is 1

Proposition (1. Young's Inequality)

Let $a, b \geq 0$ and let $p, q > 1$ be a conjugate pair. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.

First note that the inequality is trivial if $a = 0$ or $b = 0$. We assume $ab > 0$. We give two simple proofs:

1. The function e^t is convex on $\mathbb{R}$: Thus for ever $\lambda \in [0, 1]$ and for every $t, s \in \mathbb{R}$, we have
$$e^{\lambda t + (1-\lambda)s} \le \lambda e^t + (1-\lambda)e^s.$$
Young's inequality follows by taking $\lambda = \frac{1}{p}$ so that $1 - \lambda = \frac{1}{p}$ and let $t = \log(a^p)$, $s = \log(b^q)$.

2. Young's inequality is equivalent to
$$\frac{a}{b^{q-1}} \le \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}.$$
Since $q = \frac{p}{p-1} = p(q-1)$, then this inequality is equivalent to $x \le \frac{1}{p}x^p + \frac{1}{q}$ where $x = \frac{a}{b^{q-1}}$. Consider the function γ defined for $x \ge 0$ by
$$\gamma(x) = \frac{1}{p}x^p + \frac{1}{q} - x$$
We have $\gamma(0) = \frac{1}{p}$, $\lim_{x\to\infty} \gamma(x) = \infty$ $(p > 1)$ and $\gamma'(x) = x^{p-1} - 1$ vanishes only at $x = 1$ where $\gamma(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$. Therefore $\gamma(x) \ge 0$ for all $x \ge 0$.

□

Theorem (1. Hölder's Inequality)

Let $E \subset \mathbb{R}^n$ be measurable, $1 \leq p < \infty$ and let q be the conjugate of p. If $f \in L^p(E)$ and $g \in L^q(E)$, then

$$\int_E |fg|\, dx \leq \|f\|_p \, \|g\|_q$$

Proof.

If $p = 1$ and $q = \infty$, we have $|g| \leq \|g\|_\infty$ a.e. on E and therefore $\int_E |fg|\, dx \leq \|f\|_1 \, \|g\|_\infty$.

Suppose $p > 1$. If $\|f\|_p = 0$ or if $\|g\|_q = 0$, then $|fg| = 0$ a.e. on E and Hölder's inequality follows.

Now assume $\|f\|_p > 0$ and $\|g\|_q > 0$. Let $F(x) = \dfrac{f(x)}{\|f\|_p}$ and $G(x) = \dfrac{g(x)}{\|g\|_q}$. Hence $F \in L^p(E)$ with $\|F\|_p = 1$ and $G \in L^q(E)$ with $\|G\|_q = 1$. To prove Hölders inequality, it is enough to verify that $\int_E |FG|\, dx \leq 1$.

Let $Z = \{x \in E : |F(x)| = \infty \text{ or } |G(x)| = \infty\}$. Then $m(Z) = 0$. For $x \in E \backslash Z$, we can apply Young's inequality to get

$$|F(x)G(x)| \leq \frac{1}{p}|F(x)|^p + \frac{1}{q}|G(x)|^q .$$

Therefore

$$\begin{aligned}\int_E |FG|\, dx &\leq \frac{1}{p}\int_E |F|^p\, dx + \frac{1}{q}\int_E |G|^q\, dx \\ &\leq \frac{1}{p}\|F\|_p^p + \frac{1}{q}\|G\|_q^q = \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

□

A special case of Hölder's inequality is the Cauchy-Schwarz inequality: If $f, g \in L^2(E)$, then

$$\int_E |fg|\, dx \leq \|f\|_2 \, \|g\|_2 = \sqrt{\int_E |f|^2\, dx}\sqrt{\int_E |g|^2\, dx}$$

Theorem (2. Minkowski's Inequality)

Let $E \subset \mathbb{R}^n$ be measurable, $1 \leq p \leq \infty$, and $f, g \in L^p(E)$, then $f + g \in L^p$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof.

The case $p = 1$ and the case $p = \infty$ were considered earlier. Assume $1 < p < \infty$. We know that $L^p(E)$ is a vector space, hence $f + g \in L^p(E)$ if $f, g \in L^p(E)$. If $\|f + g\|_p = 0$ Minkowski is trivial. Assume $\|f + g\|_p > 0$. We can write

$$|f + g|^p = |f + g| \ |f + g|^{p-1} \leq |f| \ |f + g|^{p-1} + |g| \ |f + g|^{p-1}$$

Now we apply Hölder's inequality to $|f| \ |f + g|^{p-1}$ and to $|g| \ |f + g|^{p-1}$ to obtain

$$\int_E |f| \ |f + g|^{p-1} \, dx \leq \|f\|_p \left(\int_E |f + g|^{q(p-1)} \, dx \right)^{\frac{1}{q}} \text{ and } \int_E |g| \ |f + g|^{p-1} \, dx \leq \|g\|_p \left(\int_E |f + g|^{q(p-1)} \, dx \right)^{\frac{1}{q}}$$

where q is the conjugate of p. Since $(p - 1)q = p$ and $\frac{1}{q} = \frac{p-1}{p}$, it follows that

$$\int_E |f + g|^p \, dx \leq \left[\|f\|_p + \|g\|_p \right] \left(\int_E |f + g|^p \, dx \right)^{\frac{p-1}{p}} .$$

Minkowski's inequality follows by rearranging this inequality. □

Corollary (1)

Let $E \subset \mathbb{R}^n$ be measurable, $1 < p < \infty$, and let $\mathcal{F} \subset L^p(E)$. If the family $\mathcal{F}$ is bounded, then it is uniformly integrable over E ($\mathcal{F}$ bounded in $L^p(E)$ means that there exists $M > 0$, such that $\|f\|_p \leq M$ for all $f \in \mathcal{F}$).

Proof.

We need to prove that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for any measurable set $A \subset E$, with $m(A) < \delta$, we have $\int_A |f|\,dx < \epsilon$ for all $f \in \mathcal{F}$. Let $M > 0$ such that $\|f\|_p \leq M$ for all $f \in \mathcal{F}$. For a given $\epsilon > 0$, let $\delta = \left(\frac{\epsilon}{M}\right)^q$ where q is the conjugate of p. Let $A \subset E$, with $m(A) < \delta$. Since A has finite measure, then χ_A is in $L^q(A)$ and the restriction to A of any element in $L^p(E)$ is in $L^p(A)$. We apply Hölder inequality in the set A to $f \in L^p(E)$ and χ_A to get

$$\int_A |f|\,dx = \int_A |f|\,\chi_A\,dx \leq \|f\|_{A,p}\,\left\|\chi_A\right\|_{A,q}$$

where $\|\cdot\|_{A,p}$ denotes the norm in $L^p(A)$ to distinguish it from the norm $\|\cdot\|_p = \|\cdot\|_{E,p}$ in $L^p(E)$. We have $\left\|\chi_A\right\|_{A,q} = (m(A))^{\frac{1}{q}}$. Now for any $f \in \mathcal{F}$ we have $\|f\|_{A,p} \leq \|f\|_p \leq M$. It follows that

$$\int_A |f|\,dx \leq \|f\|_{A,p}\,(m(A))^{\frac{1}{q}} \leq M\,(m(A))^{\frac{1}{q}} < \epsilon$$

□

Corollary (2)

Let $E \subset \mathbb{R}^n$ be measurable with finite measure and let $1 \leq p_1 < p_2 \leq \infty$. Then $L^{p_2}(E) \subset L^{p_1}(E)$. Furthermore, for every $f \in L^{p_2}(E)$ we have

$$\|f\|_{p_1} \leq C\,\|f\|_{p_2}, \quad \text{with } C = \begin{cases} m(E)^{\frac{p_2-p_1}{p_1p_2}} & \text{if } p_2 < \infty \\ m(E)^{\frac{1}{p_1}} & \text{if } p_2 = \infty \end{cases}$$

Proof.

First, consider the case $p_2 = \infty$. Let $f \in L^\infty(E)$. Then there exists a set $Z \subset E$ with $m(Z) = 0$ such that $|f| \leq \|f\|_\infty$ on $E \backslash Z$. Since $m(E) < \infty$, then $\int_E |f|^{p_1}\, dz \leq \|f\|_\infty^{p_1}\, m(E)$ and so $\|f\|_{p_1} \leq C\, \|f\|_\infty$.

Next if $p_2 < \infty$, let $p = \dfrac{p_2}{p_1} > 1$ and let q be the p-conjugate. If $f \in L^{p_2}(E)$, then $|f|^{p_1} \in L^p(E)$. The function χ_E is in $L^q(E)$ (since E has finite measure). We can therefore apply Hölder inequality to the pair $|f|^{p_1}$ and χ_E to get $\int_E |f|^{p_1}\, \chi_E\, dx \leq \left\| |f|^{p_1} \right\|_p \left\| \chi_E \right\|_q$. By using $\left\| \chi_E \right\|_q = m(E)^{\frac{1}{q}} = m(E)^{\frac{p_2 - p_1}{p_2}}$, we get from the above inequality

$$\begin{aligned} \|f\|_{p_1}^{p_1} &\leq \left\| |f|^{p_1} \right\|_p \, m(E)^{\frac{p_2-p_1}{p_2}} = \left(\int_E |f|^{p_1 p} \right)^{\frac{1}{p}} m(E)^{\frac{p_2-p_1}{p_2}} \\ &\leq \left(\int_E |f|^{p_2} \right)^{\frac{p_1}{p_2}} m(E)^{\frac{p_2-p_1}{p_2}}. \end{aligned}$$

The estimate of the lemma follows by taking the p_1-root. □

Remark (1)

In general, when $m(E) = \infty$ there is no inclusion between the different $L^p(E)$ spaces (see exercises)