Real Analysis MAA 6616
Lecture 25
Convergence in L^p Spaces and Completeness

Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{f_n\}_n \subset X$ is said to converge in X if there exists $f \in X$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$ and we write $f_n \to f$ in X or $\lim_{n\to\infty} f_n = f$.

A sequence $\{f_n\}_n \subset L^\infty(E)$ converges in $L^\infty(E)$, if and only there exists $f \in L^\infty(E)$ such that $\|f_n - f\|_\infty \to 0$. This means that there exists a set of $Z \subset E$ of measure 0 such that $\{f_n\}_n$ converges uniformly to f on $E \backslash Z$.

A sequence $\{f_n\}_n \subset L^p(E)$, with $1 \le p < \infty$, converges in $L^p(E)$, if and only there exists $f \in L^p(E)$ such that $\|f_n - f\|_p \to 0$. This means that $\lim_{n\to\infty} \int_E |f_n - f|^p \, dx = 0$.

A sequence $\{f_n\}_n \subset X$ is said to be Cauchy in X if for every $\epsilon > 0$, there exists $N > 0$ such that $\|f_n - f_m\| < \epsilon$ for all $n, m > N$. A normed space $(X, \|\cdot\|)$ is said to be a Banach space or complete space if every Cauchy sequence in X is convergent.

Proposition (1)

If $\{f_n\}_n \subset X$ is a convergent sequence, then it is a Cauchy sequence. Furthermore, a Cauchy sequence in X converges if it has a convergent subsequence.

Proof.

Let $\{f_n\}_n \subset X$ be a convergent sequence (to $f \in X$) and let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon/2$ for all $n > N$. Let $n, m > N$, then $\|f_n - f_m\| \le \|f_n - f\| + \|f - f_m\| < \epsilon$ and the sequence is Cauchy.

Next, suppose $\{f_n\}_n \subset X$ is Cauchy and has a convergent subsequence $\{f_{n_j}\}_j$ with limit f. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \epsilon/2$ for all $n, m \ge N$ ($\{f_n\}_n$ is Cauchy). There exists $J \in \mathbb{N}$ such $\left\|f_{n_j} - f\right\| < \epsilon/2$ for all $j \ge J$ ($\{f_{n_j}\}_j$ converges to f). Let $K = \max(N, J)$ for $n \ge K$, we have

$$\|f_n - f\| \le \left\|f_n - f_{n_j}\right\| + \left\|f_{n_j} - f\right\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $f_n \to f$ in X. □

Theorem (1. Riesz-Fischer (Completeness of $L^p(E)$))

For $1 \leq p \leq \infty$, $L^p(E)$ is a Banach space. Furthermore if $f_n \to f$ in $L^p(E)$, then $\{f_n\}_n$ has a subsequence that converges to f pointwise a.e. on E.

Proof.

- Case $p = \infty$. Let $\{f_n\}_n \in L^\infty(E)$ be a Cauchy sequence. First note that $\{f_n\}_n$ is uniformly bounded in $L^\infty(E)$. That is, there exists $M > 0$ such that $\|f_n\|_\infty \leq M$. Indeed for $\epsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that $\|f_n - f_m\| \leq 1$ for all $n, m \geq N_1$, let $M = 1 + \max\{\|f_j\|_\infty : j = 1, \cdots, N_1\}$. Then $\|f_n\|_\infty \leq M$ for $n \leq N_1$ and for $n \geq N_1$, we have
$$\|f_n\|_\infty \leq \left\|f_n - f_{N_1}\right\|_\infty + \left\|f_{N_1}\right\|_\infty \leq 1 + \left\|f_{N_1}\right\|_\infty \leq M .$$
There exists $Z \subset E$ with $m(Z) = 0$ such that for a given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sup_{x \in E\setminus Z} |f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$. It follows that $\{f_n(x)\}_n$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in E\setminus Z$.
Therefore there exists $f(x) \in \mathbb{R}$ such that $f_n(x) \to f(x)$ for all $x \in E\setminus Z$.
Next we prove that $f \in L^\infty(E)$ and $f_n \to f$ in $L^\infty(E)$. Let $x \in E\setminus Z$, there exists $n = n(x)$ such that $|f(x) - f_n(x)| \leq 1$. Therefore
$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq 1 + \|f_n\|_\infty \leq 1 + M .$$
Let $\epsilon > 0$, and $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n, m > N$. For $x \in E\setminus Z$ we have
$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \lim_{m\to\infty} \|f_m - f_n\|_\infty \leq \epsilon.$$

- Case $1 \leq p < \infty$. Let $\{f_n\}_n \in L^p(E)$ be a Cauchy sequence. We are going to construct a convergent subsequence. For this let $k \in \mathbb{N}$, and let $\epsilon_k = \dfrac{1}{2^{k+1}}$ it follows from the Cauchy condition that there exists $N_k \in \mathbb{N}$ such that $\|f_n - f_m\| \leq \epsilon_k$ for all $n, m \geq N_k$. We can assume that the sequence of integers $\{N_k\}_k$ is strictly increasing. Set $N_0 = 0$ and $f_0 = 0$. For $j \in \mathbb{N}$ define functions g_j and h_j by
$$g_j = \sum_{k=1}^{j} (f_{N_k} - f_{N_{k-1}}) \text{ and } h_j = \sum_{k=1}^{j} \left|f_{N_k} - f_{N_{k-1}}\right|$$
Now we show that h_j converges in $L^p(E)$.

□

Proof.

CONTINUED:

It follows from Minkowski inequality that

$$\|h_j\|_p \leq \sum_{k=1}^{j} \left\|f_{N_k} - f_{N_{k-1}}\right\|_p \leq \sum_{k=1}^{j} \frac{1}{2^k} = 2 - \frac{1}{2^j}$$

Hence $\{h_j\}_j$ is a sequence of nonnegative and increasing functions in $L^p(E)$. Therefore it converges to a function h and furthermore, Fatou's Lemma implies that

$$\int_E |h|^p\, dx \leq \lim_{j\to\infty} \int_E |h_j|^p\, dx = \lim_{j\to\infty} \|h_j\|_p^p \leq 2^p\ .$$

The limit function $h = \sum_{k=1}^{\infty} \left|f_{N_k} - f_{N_{k-1}}\right|$ is therefore finite a.e. in E.

This means that the series $g = \sum_{k=1}^{\infty} (f_{N_k} - f_{N_{k-1}})$ is absolutely convergent a.e. in E. Since the series is telescoping and $f_0 = 0$, we have $g(x) = \lim_{j\to\infty} f_{N_j}(x)$. Again using Fatou's Lemma we have

$$\left\|g - f_{N_j}\right\|_p^p = \int_E \left|g - f_{N_j}\right|^p dx \leq \liminf_{i\to\infty} \int_E \left|f_{N_i} - f_{N_j}\right|^p dx \leq \liminf_{i\to\infty} \left\|f_{N_i} - f_{N_j}\right\|_p^p \leq \left(\frac{1}{2^{j+1}}\right)^p$$

Hence the subsequence $\{f_{N_j}\}_j$ of the Cauchy sequence $\{f_n\}_n$ is convergent in $L^p(E)$. Therefore $\{f_n\}_n$ is convergent (Proposition 1).

□

Remark (1)

Note that pointwise convergence does not imply convergence in L^p. For example the sequence of functions $f_n(x) = \sqrt[p]{n}\, \chi_{(0,\ 1/n]}$ converges pointwise to $f = 0$ on $(0,\ 1)$ but f_n does not converge in $L^p(0,\ 1)$ to f since $\|f_n\|_p = 1$ for all n and $\|f\|_p = 0$.

Remark (2)

The following example shows that convergence in L^p does not imply pointwise convergence a.e. In $[0,\ 1] \subset \mathbb{R}$ define a sequence f_n as follows: $f_1 = \chi_{[0,\ 1]}$, $f_2 = \chi_{[0,\ 1/2]}$, $f_3 = \chi_{[1/2,\ 1]}$. By induction suppose that for $k \in \mathbb{N}$ the functions $f_1, \cdots, f_{2^k-1}$ are defined, define $f_{2^k}, \cdots, f_{2^{k+1}-1}$ as $f_{2^k+j} = \chi_{\left[\frac{j}{2^k},\ \frac{j+1}{2^k}\right]}$ for $j = 1, \cdots, 2^k - 1$. We have $f_n \to 0$ in $L^p(0,\ 1)$ since $\left\|f_{2^k+j}\right\|_p = 2^{-k/p}$ for all $k \in \mathbb{N}$ and $j \in \{0, \cdots, 2^k - 1\}$. Next for $x \in (0,\ 1)$, and for any $k \in \mathbb{N}$ there exists a unique $j \in \{0, \cdots, 2^k - 1\}$ such that $x \in \left[\frac{j}{2^k},\ \frac{j+1}{2^k}\right]$ and then $f_{2^k+j}(x) = 1$ but $f_{2^k+j+1}(x) = 0$. This means that the sequence $f_n(x)$ is not a Cauchy sequence.

Theorem (2)

Let $E \subset \mathbb{R}^n$ be measurable and $1 \leq p < \infty$. Let $\{f_n\}_n$ in $L^p(E)$. Suppose that $f_n \to f$ a.e. on E. Then $f_n \to f$ in L^p if and only if $\lim_{n\to\infty} \int_E |f_n|^p\, dx = \int_E |f|^p\, dx$.

Proof.

There exists a set $Z \subset E$ such that $m(Z) = 0$, the f_n's and f are finite on $E_1 = E \backslash Z$ and $f_n \to f$ pointwise on E_1. It follows from the triangle inequality that $\left| \|f_n\|_p - \|f\|_p \right| \leq \|f_n - f\|_p$. Consequently if $f_n \to f$ in $L^p(E)$, then $\|f_n\|_p \to \|f\|_p$. Conversely suppose that $\|f_n\|_p \to \|f\|_p$, we need to show that $\|f_n - f\|_p \to 0$. Recall that for all $a, b \in \mathbb{R}$ we have $|a - b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. We rewrite this inequality as $\frac{|a|^p + |b|^p}{2} - \frac{|a-b|^p}{2^p} \geq 0$ so that for every $x \in E_1$ the function g_n given by

$$g_n(x) = \frac{|f_n(x)|^p + |f(x)|^p}{2} - \frac{|f_n(x) - f(x)|^p}{2^p} \qquad x \in E_1$$

is nonnegative and moreover, $g_n \to |f|^p$ pointwise on E_1. Now Fatou's Lemma implies that

$$\int_{E_1} |f^p|\, dx \leq \liminf_{n\to\infty} \int_{E_1} g_n dx = \int_{E_1} |f|^p\, dx - \limsup_{n\to\infty} \int_{E_1} \frac{|f_n - f|^p}{2^p} dx\,.$$

Consequently $\limsup_{n\to\infty} \int_{E_1} |f_n - f|^p dx = 0$ and $\|f_n - f\|_p \to 0$. $\square$

Let $(X, \|\cdot\|)$ be a normed space and let $\mathcal{F}$ and $\mathcal{G}$ be such that $\mathcal{F} \subset \mathcal{G} \subset X$. The family $\mathcal{F}$ is said to be dense in $\mathcal{G}$ if for every $g \in \mathcal{G}$ and for every $\epsilon > 0$ there exists $f \in \mathcal{F}$ such that $\|f - g\| < \epsilon$. This is equivalent to saying $\mathcal{F}$ dense in $\mathcal{G}$ if and only if for every $g \in \mathcal{G}$ there exists a sequence $\{f_n\}_n \subset \mathcal{F}$ such that $\|f_n - g\| \to 0$. This means that the closure of $\mathcal{F}$ is the closure of $\mathcal{G}$.
Note that if $\mathcal{F} \subset \mathcal{G} \subset \mathcal{H} \subset X$, then $\mathcal{F}$ dense in $\mathcal{G}$ and $\mathcal{G}$ dense in $\mathcal{H}$ implies $\mathcal{F}$ dense in $\mathcal{H}$. Also $\mathcal{F}$ dense in $\mathcal{H}$ implies $\mathcal{G}$ dense in $\mathcal{H}$

Let $E \subset \mathbb{R}^n$ be measurable. A function $f \in L^p(E)$ is said to have compact support if there exists a compact set $K \subset \mathbb{R}^q$ such that $f = 0$ a.e. on $E \setminus K$. Denote by $L^p_c(E)$ the space of functions in $L^p(E)$ with compact support.

Theorem (3)

$L^p_c(E)$ is dense in $L^p(E)$.

Proof.

Let $f \in L^p(E)$, we need to show that there exists a sequence $\{f_j\} \subset L^p_c(E)$ such that $\|f - f_j\|_p \to 0$ as $j \to \infty$. Let C_j be the cube in $\mathbb{R}^n$ centered at 0 and with side length $2j$: $C_j = [-j, j]^n$. Define f_j by $f_j = f\chi_{C_j \cap E}$. Then f_j is compactly supported with support C_j and so $f_j \in L^p_c(E)$. Furthermore, $f_j \to f$ pointwise on E and $|f - f_j|^p = \left|f\chi_{E\setminus C_j}\right|^p \leq |f|^p \in \mathcal{L}(E)$. It follows from the Dominated Convergence Theorem that $\int_E |f - f_j|^p \to 0$. That is f_j converges to f in $L^p(E)$. □

Proposition (1)

Let $E \subset \mathbb{R}^n$ be measurable, $1 \leq p \leq \infty$, and let $\mathcal{S}^p(E)$ be the family of simple functions in $L^p(E)$. Then $\mathcal{S}^p(E)$ is dense in $L^p(E)$.

Proof.

First consider the case $p = \infty$. Let $f \in L^\infty(E)$. There exists a set $Z \subset E$ with $m(Z) = 0$ and $M > 0$ such that $|f| \leq M$ on $E_1 = E \backslash Z$. Given $\epsilon > 0$, there exists a simple function ϕ on E_1 such that $|\phi - f| < \epsilon$ (Simple Approximation Lemma). Therefore $\mathcal{S}^\infty(E)$ is dense in $L^\infty(E)$.
Next suppose that $1 \leq p < \infty$. Let $f \in L^p(E)$. Then f is measurable on E and so there exists a sequence $\{\phi_n\}_n$ of simple functions on E such that $\phi_n \to f$ pointwise a.e on E and $|\phi_n| \leq |f|$ on E for all n (Simple Approximation Theorem). It follows that, $\int_E |\phi_n|^p \, dx \leq \int_E |f|^p \, dx$. This means $\{\phi_n\}_n \subset \mathcal{S}^p(E)$. We are left to verify that $\phi_n \to f$ in $L^p(E)$.
We have $|\phi_n - f|^p \to 0$ pointwise on E. Moreover,

$$|\phi_n - f|^p \leq 2^p \left(|\phi_n|^p + |f|^p\right) \leq 2^{p+1} |f|^p .$$

It follows from Lebesgue Dominated Convergence Theorem that $\lim_{n\to\infty} \int_E |\phi_n - f|^p \, dx = 0$. □

A normed space $(X, \|\cdot\|)$ is said to be separable if there exists $\mathcal{C} \subset X$ such that $\mathcal{C}$ is countable and dense in X. For example $\mathbb{R}^n$ is separable since $\mathbb{Q}^n$ is countable is dense in $\mathbb{R}^n$.

Theorem (4)

Let $E \subset \mathbb{R}^n$ be a measurable and $1 \leq p < \infty$. Then $L^p(E)$ is separable.

Proof.

First we prove that $L^p(\mathbb{R}^n)$ is separable. Consider the dyadic decomposition of $\mathbb{R}^n$: Let $C = [0, \ 1)^n$ be the unit cube of $\mathbb{R}^n$. For $k \in \mathbb{N}$ and $J \in \mathbb{Z}^n$, let $C_{k,J} = \dfrac{J + C}{2^k}$ (the cube with a side length $1/2^k$ and a vertex at $J/2^k$). Then for every k, $\mathbb{R}^n = \bigcup_{k \in \mathbb{N}, \ J \in \mathbb{Z}^n} C_{k,J}$. The set $S = \{(J, k) \ : \ J \in \mathbb{Z}^n, \ k \in \mathbb{N}\}$ is countable and for every open set $U \in \mathbb{R}^n$ there exists a set $I \subset S$ such that $U = \bigcup_{(J,k) \in I} C_{k,J}$.

Let $\mathcal{S}_{\mathbb{Q}}(\mathbb{R}^n)$ be the collection of all (finite) linear combinations of characteristic functions of the dyadic cubes with coefficients in $\mathbb{Q}$. Thus a simple function ϕ is in $\mathcal{S}_{\mathbb{Q}}(\mathbb{R}^n)$ if and only if there exist $(J_1, k_1), \cdots, (J_s, k_s) \in S$ and $r_1, \cdots, r_s \in \mathbb{Q}$ such that $\phi = \sum_{j=1}^{s} r_j \chi_{C_{k_j, J_j}}$. Similarly consider the family $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$ of all (finite) linear combinations of characteristic functions of the dyadic cubes with coefficients in $\mathbb{R}$. The family $\mathcal{S}_{\mathbb{Q}}(\mathbb{R}^n)$ is countable and $\mathcal{S}_{\mathbb{Q}}(\mathbb{R}^n) \subset \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. It follows from the density of $\mathbb{Q}$ in $\mathbb{R}$ that $\mathcal{S}_{\mathbb{Q}}(\mathbb{R}^n)$ is dense in $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$.

Let $f \in L^p(\mathbb{R}^n)$. Given $\epsilon > 0$, we know (Proposition 1) that there exists a simple function ψ such that $\|f - \psi\|_p < \epsilon$. Thus to prove the density of $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ it is enough to prove that given a simple function ψ and $\epsilon > 0$, there exists $\phi \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$ such that $\|\phi - \psi\|_p < \epsilon$. Since $\psi = \sum_{j=1}^{N} a_j \chi_{A_j}$ for some disjoint measurable sets $A_1, \cdots, A_N \subset \mathbb{R}^n$, then to prove the density of $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$ in the space simple functions in $L^p(\mathbb{R}^n)$ it is enough to prove that given any measurable set $A \subset \mathbb{R}^n$ and $\epsilon > 0$, there exists a $\phi \in \mathcal{S}_{\mathbb{R}}(\mathbb{R}^n)$ such that $\left\|\phi - \chi_A\right\|_p < \epsilon$.

□

Proof.

CONTINUED: Given $A \subset \mathbb{R}^n$ measurable with A bounded. We can find an open set $U \supset A$ such that $m(U \backslash A) < \epsilon$. There exist a finite set $I \subset S$ such that $\bigcup_{(J,k)\in I} C_{k,J} \subset U$ and $m(U) - \sum_{(J,k)\in I} \mathrm{vol}(C_{k,J}) < \epsilon$. The function

$$\phi = \sum_{(J,k)\in I} \chi_{C_{k,J}}$$

is in $\mathcal{S}_{\mathbb{Q}}(\mathbb{R}^n)$ and

$$\left\|\chi_A - \phi\right\|_p \leq \left\|\chi_A - \chi_U\right\|_p + \left\|\chi_U - \phi\right\|_p \leq m(U\backslash A)^{1/p} + \left[m(U) - \sum_{(J,k)\in I} \mathrm{vol}(C_{k,J})\right]^{1/p} \leq 2\epsilon^{1/p}$$

If A is unbounded and $m(A) < \infty$. Write $A = \bigcup_{i\in\mathbb{N}} A_i$ where $A_i = A \cap B(0, i)$, where $B(0, i)$ is the ball centered at 0 with radius i and apply the preceding result to A_i. □

Remark (2)

In general the space $L^\infty(E)$ is not separable.

Example

Let $a, b \in \mathbb{R}$ with $a < b$ we are going to prove that $L^\infty[a,\ b]$ is not separabale. By contradiction, suppose that it is separable. Then there would exists sequence $\{f_n\}_{n\in\mathbb{N}}$ that is dense in $L^\infty[a,\ b]$. For every $x \in [a,\ b]$ we can find an integer $n(x) \in \mathbb{N}$ such that $\left\|\chi_{[a,\ x]} - f_{n(x)}\right\|_\infty < 1/2$.

Let $a < x_1 < x_2 < b$, then $\left\|\chi_{[a,\ x_1]} - \chi_{[a,\ x_2]}\right\|_\infty = 1$. Hence

$$\begin{aligned}\|f_{n(x_1)} - f_{n(x_2)}\|_\infty &= \left\|f_{n(x_1)} - \chi_{[a,\ x_1]} + \chi_{[a,\ x_1]} - \chi_{[a,\ x_2]} + \chi_{[a,\ x_2]} - f_{n(x_2)}\right\|_\infty \\ &\geq \left\|\chi_{[a,\ x_1]} - \chi_{[a,\ x_2]}\right\|_\infty - \left\|f_{n(x_1)} - \chi_{[a,\ x_1]}\right\|_\infty - \left\|\chi_{[a,\ x_2]} - f_{n(x_2)}\right\|_\infty \\ &> 0\end{aligned}$$

This means $f_{n(x_1)} \neq f_{n(x_2)}$ and the map $[a,\ b] \longrightarrow \mathbb{N}$ given by $x \to n(x)$ is one to one. This is a contradiction since $[a,\ b]$ is uncountable. Therefore $L^\infty[a,\ b]$ is not separable.