Real Analysis MAA 6616
Lecture 26
Bounded Linear Functionals on L^p
Weak Convergence

Let $(X, \|\cdot\|)$ be a normed space. A linear functional on X is a map
$T : X \longrightarrow \mathbb{R}$ such that $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$.

Note that if T and S are two linear functional on X, and $a, b \in \mathbb{R}$, the $aT + bS$ defined on X by $(aT + bS)(f) = aT(f) + bS(f)$ is again a linear functional on X. Thus the set of all linear functionals on X is a linear space.

The linear functional T is said to be bounded if there exists $M > 0$ such that $|T(f)| \leq M \|f\|$ for all $f \in X$. Denote by X^* the space of all bounded linear functionals on X. the space X^* is called the dual of X.
For linear functional $T \in X^*$, define $\|T\|_*$ by
$$\|T\|_* = \inf\{M : |Tf| \leq M \|f\| \text{ for all } f \in X.\} = \sup\{|Tf| : f \in X \text{ with } \|f\| \leq 1.\}$$

Theorem (1)

$(X^*, \|\cdot\|_*)$ is a normed space. Moreover, if X is a Banach space, then so is X^*.

Proof.

The verification that $\|\cdot\|_*$ is a norm is left as an exercise. Next we very that X^* is Banach when X is Banach. Let $\{T_n\}$ be a Cauchy sequence in X^*. Thus for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|T_n - T_m\|_* < \epsilon$ for all $n, m > N$. Hence for any given $f \in X$, we have $|T_n f - T_m f| < \epsilon \|f\|$. This implies that $\{T_n f\}$ is a Cauchy sequence in $\mathbb{R}$. Therefore $T_n f \to T(f)$ for some $T(f) \in \mathbb{R}$. It follows from the linearity of T_n that $T_n(\alpha f + \beta g)) \to \alpha T(f) + \beta T(g)$ so that $T = \lim_{n\to\infty} T_n$ is a linear operator on X. Next we need to verify that T is bounded and $\|T - T_n\|_* \to 0$.

Since the sequence $\{T_n\}$ is Cauchy, then it is uniformly bounded (by M). It follows that for $f \in X$, we have

$|Tf| \leq \lim_{n\to\infty} [|Tf - T_n f| + |T_n f|] \leq \lim_{n\to\infty} [|Tf - T_n f| + M \|f\|] \leq M \|f\|$. Hence $T \in X^*$. Finally, for

$\epsilon > 0$ there exists $f \in X$ with $\|f\| \leq 1$, such that $\|T - T_n\|_* \leq |Tf - T_n f| + \epsilon$. Since $|Tf - T_n f| \to 0$, then

$\|T - T_n\|_* \to 0$. □

Let $E \subset \mathbb{R}^n$ measurable. For $1 \le p < \infty$, the dual space of $L^p(E)$ is the space

$$L^p(E)^* = \{T : L^p(E) \longrightarrow \mathbb{R} : \ T \text{ linear and bounded}\}$$

with norm

$$\|T\|_* = \inf\left\{M : \ |Tf| \le M\ \|f\|_p \ \text{ for all } f \in L^p(E).\right\} = \sup\{|Tf| : f \in L^p(E) \text{ with } \|f\| \le 1.\}$$

Theorem (2)

Let $1 \le p < \infty$. Then $L^p(E)^* \cong L^q(E)$, where q is the conjugate of p.

The theorem says that the dual of $L^p(E)$ can be identified through an isometry with the space $L^q(E)$ where $1/p + 1/q = 1$. This is an important theorem in analysis whose proof will be postponed until we develop the necessary tools of abstract measure theory and prove another result: Radon-Nikodym Theorem. For now we take a closer look at bounded functionals on L^p.

For a real number a define $\operatorname{sgn}(a) = 1$ if $a > 0$, $\operatorname{sgn}(a) = 0$ if $a = 0$, and $\operatorname{sgn}(a) = -1$ if $a < 0$.

Lemma (1)

Let $1 \le q < \infty$ and $g \in L^q(E)$ with $g \ne 0$, then the function $g^* = \|g\|_q^{1-q} \operatorname{sgn}(g)\ |g|^{q-1}$ is in $L^p(E)$ with $p^{-1} + q^{-1} = 1$ and $\|g^*\|_p = 1$.

Proof.

Use the relation $p(q-1) = q$ to get

$$\|g^*\|_p^p = \|g\|_q^{p(1-q)} \int_E |g|^{p(q-1)} = \|g\|_q^{p(1-q)}\ \|g\|_q^{p(q-1)} = 1$$

□

Theorem (3)

Let $1 < p \leq \infty$ *and* $1 \leq q < \infty$ *with* $p^{-1} + q^{-1} = 1$. *Let* $g \in L^q(E)$. *Consider the map* $T : L^p(E) \longrightarrow \mathbb{R}$ *given by*

$$Tf = \int_E fgdm \, .$$

Then $T \in L^p(E)^*$ *and* $\|T\|_* = \|g\|_q$.

Proof.

The linearity of T follows from the linearity of the integral. The boundedness of T follows from Hölder inequality:

$$|Tf| = \left| \int_E fgdm \right| \leq \int_E |f| \; |g| \, dm \leq \|g\|_q \, \|f\|_p \, .$$

So $T \in L^p(E)^*$ and $\|T\|_* \leq \|g\|_q$. It remains to show that $\|g\|_q \leq \|T\|_*$.

For this, we first consider the case $p = \infty$ so that $q = 1$. Let $f = \operatorname{sgn}(g)$ ($\|f\|_\infty = 1$). Then

$T(f) = \int_E |g| \, dm = \|g\|_1 \; \|f\|_\infty$. This means that $\|g\|_1 \leq \|T\|_*$ and the theorem is proved in this case.

When $p < \infty$. Let $f = g^*$, where g^* is the function given in Lemma 1. Then $f \in L^p(E)$ and $\|f\|_p = 1$. We have

$$|Tf| = \left| \int_E g^* gdm \right| = \int_E \|g\|_q^{1-q} \, |g|^q = \|g\|_q^{1-q} \; \|g\|_q^q = \|g\|_q \; \|f\|_p$$

Therefore $\|g\|_q \leq \|T\|_*$. □

The next step is to show that if $T \in L^p(E)^*$, then there exists a unique element $g \in L^q(E)$ such that T is given by

$$Tf = \int_E fg \, dm \;\; \text{for all} \; f \in L^p(E) \, .$$

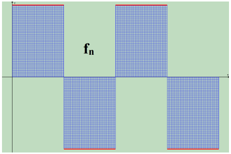
This is known as the Riesz representation.

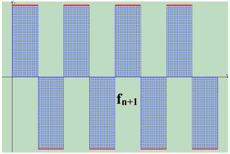
Recall that the Bolzano Weierstrass Theorem states if $\{x_j\}$ is a bounded sequence in $\mathbb{R}^n$, then it has a convergent subsequence. This result does not extend to infinite dimensional Banach spaces. In particular there exist bounded sequences in L^p spaces that do not have convergent subsequences.

Example

Consider the sequence $\{f_n\}$ defined in $[0,\ 1]$ by

$$f_n(x) = (-1)^j \text{ for } \frac{j}{2^n} \leq x < \frac{j+1}{2^n} \text{ with } j = 0, 1, \cdots, 2^n - 1$$





For $1 \leq p \leq \infty$, each function f_n is in $L^p([0,\ 1])$ and moreover, $\|f_n\|_p = 1$ for all n (since $|f_n(x)| = 1$ for all x. Therefore $\{f_n\}$ is a bounded sequence in $L^p([0,\ 1])$.
Now consider two positive integers $n > m$. Write $n = m + r$ with $r > 0$. Let $j \in \{0, \cdots, 2^m - 1\}$. The function f_m is constant on the interval $I_j = \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right)$ ($f_m = (-1)^j$). In the interval I_j, there are 2^r intervals $J_{j,k} = \left[\frac{j+k}{2^{m+r}}, \frac{j+k+1}{2^{m+r}}\right)$ in each of which f_{m+r} is constant $f_{m+r} = (-1)^{j+k}$ on $J_{j,k}$. Therefore $|f_m - f_{m+r}| = 2$ on the union of 2^{r-1} such intervals with total measure $\frac{2^{r-1}}{2^{m+r}} = 2^{-(m+1)}$. Since there are 2^m such intervals I_j, this means that $|f_m - f_{m+r}| = 2$ on a union of intervals with total length 2^{-1}. Consequently, $\|f_n - f_m\|_p = 2^{1-\frac{1}{p}}$. This implies that the sequence does not have any Cauchy subsequence in $L^p([0,\ 1])$.

Let $(X, \|\cdot\|)$ be a normed space. A sequence $\{f_n\} \subset X$ is said to converge weakly to f in X if for every $T \in X^*$ the sequence $\{Tf_n\} \subset \mathbb{R}$ converges to Tf ($\lim_{n\to\infty} Tf_n = Tf$). In this case we write $f_n \rightharpoonup f$ in X.

This mode of convergence is different from the strong convergence $f_n \to f$ in X which means $\lim_{n\to\infty} \|f - f_n\| = 0$.

Note that

$$[f_n \to f \ \text{ in } X\,] \implies [f_n \rightharpoonup f \ \text{ in } X\,]$$

Indeed, suppose $f_n \to f$ in X. Let $T \in X^*$, then

$$|Tf_n - Tf| = |T(f_n - f)| \le \|T\|_* \, \|f_n - f\| \to 0.$$

Therefore $f_n \rightharpoonup f$. The converse is not true.

Since for for a measurable set E and $1 \le p < \infty$ the dual space of $L^p(E)$ is identified with $L^q(E)$ where q is the conjugate of p (i.e. $L^p(E)^* \cong L^q(E)$ to be established), then we have

Proposition (1)

Let $\{f_n\} \subset L^p(E)$. Then $f_n \rightharpoonup f$ in $L^p(E)$ if and only if for every $g \in L^q(E)$ (where $p^{-1} + q^{-1} = 1$) we have

$$\lim_{n\to\infty} \int_E g f_n \, dm = \int_E g f \, dm$$

Proposition (2)

A sequence $\{f_n\} \subset L^p(E)$ can converges weakly to at most one limit $f \in L^p(E)$.

Proof.

Suppose that $\{f_n\} \subset L^p(E)$ is such that $f_n \rightharpoonup f^1$ and $f_n \rightharpoonup f^2$, we need to show $f^1 = f^2$. Let $g = (f^1 - f^2)^* = \left\|f^1 - f^2\right\|_p^{1-p} \operatorname{sign}(f^1 - f^2) \left|f^1 - f^2\right|^{p-1}$ (considered in Lemma 1). Then $g \in L^q(E)$ and $\|g\|_q = 1$, where q is the conjugate of p. We have

$$\int_E gf^1 \, dm = \lim_{n\to\infty} \int_E gf_n \, dm = \int_E gf^2 \, dm .$$

As a consequence we have $0 = \int_E g(f^1 - f^2) \, dm = \int_E (f^1 - f^2)^* (f^1 - f^2) \, dm = \left\|f^1 - f^2\right\|_p$. Therefore $f^1 = f^2$. □

Theorem (4)

If $\{f_n\} \subset L^p(E)$ converges weakly to f, then $\{f_n\}$ is bounded and $\|f\|_p \le \liminf_{n\to\infty} \|f_n\|_p$.

Proof.

We start by proving the inequality. Let q be the conjugate of p and $f^* \in L^q$ be as in Lemma 1. Then $\left\|f^*\right\|_q = 1$ and it follows from Hölder's inequality that $\left|\int_E f^* f_n dm\right| \le \left\|f^*\right\|_q \|f_n\|_p = \|f_n\|_p$. It follows from this and the weak convergence of $\{f_n\}$ that

$$\|f\|_p = \int_E f^* f \, dm = \lim_{n\to\infty} \int_E f^* f_n dm \le \liminf_{n\to\infty} \|f_n\|_p \ .$$

The boundedness of the sequence will be proved by contradiction. Suppose that $\{\|f_n\|_p\}_n$ is unbounded. In this case we are going to show that we can assume without loss of generality that $\|f_n\|_p = n3^n$ for all n. This will be achieved by replacing (if necessary) the initially given sequence $\{f_n\}$ by subsequence. □

Proof.

CONTINUED: Since our assumption is $\{\|f_n\|_p\}_n$ is unbounded, then there exists n_1 such that $\left\|f_{n_1}\right\|_p \geq 3$. Let n_2 be the first integer $> n_1$ such that $\left\|f_{n_2}\right\|_p \geq 2 \cdot 3^2$. By induction, suppose that we have $n_1 < n_2 < \cdots < n_j$ such that $\left\|f_{n_k}\right\|_p \geq k3^k$ for $k = 1, \cdots, j$. Let n_{j+1} be the first integer $> n_j$ such that $\left\|f_{n_{j+1}}\right\|_p \geq (j+1)3^{j+1}$. Hence we can assume (after replacing $\{f_n\}_n$ by its subsequence $\{f_{n_j}\}_j$) that $\|f_n\|_p \geq n3^n$ for all n.

Now let $r_n = \dfrac{f_n}{n3^n}$ then $\|r_n\|_p \geq 1$ for all n. If $\{\|r_n\|_p\}_n$ is bounded, then we can find a subsequence $\{r_{n_j}\}_j$ such that $\left\|r_{n_j}\right\|_p$ converges to a limit $\alpha \geq 1$. If $\{\|r_n\|_p\}_n$ is unbounded, then we can find a subsequence $\{r_{n_j}\}_j$ that converges to ∞. In both cases we have a subsequence $\{r_{n_j}\}_j$ such that $\left\|r_{n_j}\right\|_p \to \alpha$ with $\alpha \in [1, \infty]$. This means that we can assume that $\left\|\dfrac{f_n}{n3^n}\right\|_p$ converges to $\alpha \in [1, \infty]$.

Next, let $s_n = \dfrac{n3^n}{\|f_n\|_p} f_n$. Then $\|s_n\|_p = n3^n$. Moreover, for any $g \in L^q(E)$ we have

$$\int_E s_n g \, dm = \frac{n3^n}{\|f_n\|_p} \int_E f_n g \, dm \longrightarrow \frac{1}{\alpha} \int_E fg \, dm$$

This means $s_n \rightharpoonup f/\alpha$.

After this reduction, we are now in a situation where $\|f_n\|_p = n3^n$ and $f_n \rightharpoonup f$. For each n let $f_n^* \in L^q(E)$ be the function defined in Lemma 1 so that $\|f_n^*\|_q = 1$. Define the sequence of real numbers $\{\beta_k\}$ as follows: $\beta_1 = \dfrac{1}{3}$; $\beta_2 = \dfrac{1}{3^2}$ if $\int_E f_1^* f_2 dm \geq 0$ and $\beta_2 = \dfrac{-1}{3^2}$ if $\int_E f_1^* f_2 dm < 0$. In general, suppose that $\beta_1, \cdots, \beta_n$ are defined, we define β_{n+1} as

$$\beta_{n+1} \begin{cases} \dfrac{1}{3^{n+1}} & \text{if } \displaystyle\int_E \left[\sum_{j=1}^n \beta_j f_j^*\right] f_{n+1} dm \geq 0; \\[2ex] \dfrac{-1}{3^{n+1}} & \text{if } \displaystyle\int_E \left[\sum_{j=1}^n \beta_j f_j^*\right] f_{n+1} dm < 0. \end{cases}$$

Proof.

CONTINUED: Note that since $\int_E f_n^* f_n dm = \|f_n\|_p = n3^n$ and since β_n and $\int_E \left[\sum_{j=1}^{n-1} \beta_j f_j^*\right] f_n dm$ have the same sign, then

$$\left|\int_E \left[\sum_{j=1}^{n} \beta_j f_j^*\right] f_n dm\right| = \left|\int_E \left[\sum_{j=1}^{n-1} \beta_j f_j^*\right] f_n dm + \int_E \beta_n f_n^* f_n dm\right| \geq \frac{1}{3^n} \|f_n\|_p = n$$

Consider the sequence in $L^q(E)$ given by $g_n = \sum_{j=1}^{n} \beta_j f_j^*$. We have $\left\|\beta_j f_j^*\right\|_q = 3^{-j}$. Hence for $n = m + k > m$, we have

$$\|g_n - g_m\|_q = \left\|\sum_{j=1}^{k} \beta_{m+j} f_{m+j}^*\right\|_q \leq \sum_{j=1}^{k} \left\|\beta_{m+j} f_{m+j}^*\right\|_p = \sum_{j=1}^{k} \frac{1}{3^{m+j}} \leq \frac{1}{3^m}$$

This means that the sequence $\{g_n\}$ is a Cauchy sequence in the Banach space $L^q(E)$. Hence

$g_n = \sum_{j=1}^{n} \beta_j f_j^* \longrightarrow g = \sum_{j=1}^{\infty} \beta_j f_j^* \in L^q(E)$.

Next, we use the triangle inequality, Hölder inequality, together with $\|f_n\|_p = n3^n$ to obtain

$$\begin{aligned}\left|\int_E g f_n \, dm\right| &= \left|\int_E \left[\sum_{j=1}^{\infty} \beta_j f_j^*\right] f_n dm\right| \geq \left|\int_E \left[\sum_{j=1}^{n} \beta_j f_j^*\right] f_n dm\right| - \left|\int_E \left[\sum_{j=n+1}^{\infty} \beta_j f_j^*\right] f_n dm\right| \\ &\geq n - \left(\sum_{j=n+1}^{\infty} \frac{1}{3^j}\right) \|f_n\|_p = n - \frac{1}{3^{n+1}} \left(\sum_{k=0}^{\infty} \frac{1}{3^k}\right) n3^n \\ &\geq \frac{n}{2}\end{aligned}$$

This implies that $\lim_{n\to\infty} \int_E g f_n dm \neq \int_E g f dm$ and this contradicts $f_n \rightharpoonup f$. Conclusion the sequence $\{f_n\}$ is bounded in $L^p(E)$ □