Real Analysis MAA 6616
Lecture 27
Weak Convergence: Some Consequences

Proposition (1)

Let $E \subset \mathbb{R}^n$ be measurable, $1 \leq p < \infty$ and let q be the conjugate of p. If $\{f_n\}_n$ is a weakly convergent sequence to f in $L^p(E)$ and $\{g_n\}_n$ is a strongly convergent sequence to g in $L^q(E)$ (i.e $f_n \rightharpoonup f$ in L^p and $g_n \to g$ in L^q), then

$$\lim_{n\to\infty} \int_E f_n g_n dm = \int_E fg dm.$$

Proof.

Let $\epsilon > 0$. Since $f_n \rightharpoonup f$, then $\{f_n\}$ is bounded. Let $M > 0$ such that $\|f_n\|_p \leq M$ for all n. It follows from the weak convergence of f_n that $\lim_{n\to\infty} \int_E gf_n = \int_E gf$ and therefore there exists $N_1 > 0$ such that

$$\left| \int_E gf_n - \int_E gf \right| \leq \frac{\epsilon}{2} \quad \text{for all } \; n \geq N_1 .$$

The sequence $\{g_n\}$ converges strongly to g in $L^q(E)$ implies that there exists $N_2 > 0$ such that $\|g_n - g\|_q < \dfrac{\epsilon}{2M}$ for all $n \geq N_2$.

Let $N = \max(N_1, N_2)$. For $n > N$, we have

$$\begin{aligned} \left| \int_E f_n g_n dm - \int_E fg dm \right| &= \left| \int_E f_n(g_n - g)dm + \int_E (f_n - f)g dm \right| \leq \left| \int_E f_n(g_n - g)dm \right| + \left| \int_E (f_n - f)g dm \right| \\ &\leq \|f_n\|_p \; \|g_n - g\|_q + \frac{\epsilon}{2} \leq M\frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

Let X be a linear space. A subset $Y \subset X$ is said to be the linear span of a subset $\mathcal{F} \subset X$ if Y is generated by finite linear combination of elements in $\mathcal{F}$. That is, for every $y \in Y$, there exists $f_1, \cdots, f_n \in \mathcal{F}$ and $a_1, \cdots, a_n \in \mathbb{R}$ such that

$$y = \sum_{j=1}^{n} a_j f_j$$

In this case we write $Y = \text{linspan}(\mathcal{F})$.

Proposition (2)

Let $E \subset \mathbb{R}^n$ measurable, $1 \leq p < \infty$, and q be the conjugate of p. Let $\mathcal{F} \subset L^q(E)$ be such that $\operatorname{linspan}(\mathcal{F})$ *is dense in $L^q(E)$. A bounded sequence $\{f_n\}_n \subset L^p(E)$ converges weakly to f if and only if*

$$\lim_{n\to\infty} \int_E f_n g \, dm = \int_E fg \, dm \text{ for all } g \in \mathcal{F}$$

Proof.

"$\Longleftarrow$" This implication is clear since the condition in proposition holds for all $g \in L^q(E)$ and $\operatorname{linspan}(\mathcal{F}) \subset L^q(E)$.

"$\Longrightarrow$" We need to show that $\lim_{n\to\infty} \int_E (f_n - f)h \, dm = 0$ for all $h \in L^q(E)$. Provided that the limit holds for all $g \in \operatorname{linspan}(\mathcal{F})$.

Let $h \in L^q(E)$ and $\epsilon > 0$. Let M be an upper bound for $\{\|f_n\|_p\}_n$. Since $\operatorname{linspan}(\mathcal{F})$ is dense in $L^q(E)$, then there is $g \in \operatorname{linspan}(\mathcal{F})$ such that $\|g - h\|_q \leq \dfrac{\epsilon}{2(M + \|f\|_p)}$. For such $g \in \operatorname{linspan}(\mathcal{F})$ there exists $N \in \mathbb{N}$ such that

$\left|\int_E (f_n - f)g \, dm\right| < \dfrac{\epsilon}{2}$ for all $n > N$.

It follows that for $n > N$ we have

$$\begin{aligned} \left|\int_E (f_n - f)h \, dm\right| &\leq \left|\int_E (f_n - f)(h - g) \, dm\right| + \left|\int_E (f_n - f)g \, dm\right| \\ &\leq \|f_n - f\|_p \, \|h - g\|_q + \frac{\epsilon}{2} \leq (\|f_n\|_p + \|f\|_p) \frac{\epsilon}{2(M + \|f\|_p)} + \frac{\epsilon}{2} \\ &\leq \epsilon \end{aligned}$$

$\square$

As an application we have the following characterization of weak convergence.

Theorem (1)

Let $E \subset \mathbb{R}^n$ be measurable and $1 \leq p < \infty$. A bounded sequence $\{f_n\}_n \subset L^p(E)$ converges weakly to $f \in L^p(E)$ if and only if for every measurable set $A \subset E$ we have

$$\lim_{n\to\infty} \int_A f_n \, dm = \int_A f \, dm.$$

Moreover, when $p > 1$ it is enough to consider only the subsets A with finite measure.

Proof.

Let $\mathcal{M}(E)$ be the collection of measurable subsets in E and let $\mathcal{S}(E)$ be the collection characteristic functions on subsets in $\mathcal{M}(E)$:

$$\mathcal{S}(E) = \left\{\chi_A \,:\, A \in \mathcal{M}(E)\right\}.$$

Then $\text{linspan}(\mathcal{S}(E))$ is the space of simple functions and we know that it is dense in $L^q(E)$. Hence it follows from the previous proposition that a bounded sequence $\{f_n\}_n \subset L^p(E)$ converges weakly to $f \in L^p(E)$ if and only for every $A \in \mathcal{M}(E)$ we have

$$\lim_{n\to\infty} \int_E f_n \chi_A \, dm = \int_E f\chi_A \, dm \iff \lim_{n\to\infty} \int_A f_n \, dm = \int_A f \, dm.$$

When $p > 1$, the space of simple functions with finite support is dense in $L^p(E)$. □

Consider an interval $[a, b] \subset \mathbb{R}$. The collection of step functions $\chi_{[a,c]}$ with $a \leq c \leq b$ generates (through linear combinations) the space of simple functions on $[a, b]$. Then as a consequence of the above theorem we have.

Corollary (1)

Let $1 < p < \infty$. A bounded sequence $\{f_n\}_n \subset L^p[a, b]$ converges weakly to $f \in L^p[a, b]$ if and only if

$$\lim_{n\to\infty} \int_a^c f_n dx = \int_0^c f dx \;\; \text{for all } c \in [a, b]$$

Example

(Riemann-Lebesgue Lemma) Consider the sequence of functions $f_n(x) = \sin nx$ on the interval $I = [-\pi, \pi]$. We have $|f_n| \leq 1$ for all n. Hence $\{f_n\}_n$ is a bounded sequence in the space $L^p(I)$. For $c \in I$, we have

$$\left| \int_{-\pi}^{c} f_n(x)dx \right| = \left| \left[\frac{-\cos nx}{n} \right]_{-\pi}^{c} \right| = \left| \frac{(-1)^n - \cos nc}{n} \right| \leq \frac{2}{n} \longrightarrow 0.$$

It follows from Corollary 1 that $\sin(nx) \rightharpoonup 0$ in $L^p(I)$.
Now we prove that **no** subsequence of $\{\sin(nx)\}_n$ converge to 0 in $L^p(I)$. We use trigonometric identities to get

$$\begin{aligned} \|\sin(nx)\|_p^p &= \int_{-\pi}^{\pi} |\sin(nx)|^p \, dx = 2\int_0^{\pi} |\sin(nx)|^p \, dx = \frac{2}{n}\int_0^{n\pi} |\sin t|^p \, dt \\ &= 2\int_0^{\pi} \sin t^p dt = 4\int_0^{\pi/2} \sin t^p dt = 4\int_0^{\pi/2} \sin t \left(1 - \cos^2 t\right)^{\frac{p-1}{2}} dt \\ &= 4\int_0^1 \left(1 - s^2\right)^{\frac{p-1}{2}} ds = 4\left(1 - \tau^2\right)^{\frac{p-1}{2}} \quad \text{for some } \tau \in (0, 1) \end{aligned}$$

Hence $\|\sin(nx)\|_p = 4^{\frac{1}{p}} \left(1 - \tau^2\right)^{\frac{p-1}{2p}}$ for all n. It follows that no subsequence can converge to 0.

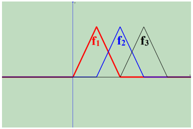
The next two examples show that a given sequence can converge pointwise but does not converge weakly in L^1. However the sequence converge weakly in L^p for $p > 1$.

Example

Let $\{f_n\}_n \subset L^1[0, 1]$ be given by $f_n = n\chi_{[0, 1/n]}$. We have $\|f_n\|_1 = 1$ for all n. Moreover, f_n converges pointwise to $f = 0$. However, f_n does not converge weakly to f since if we take $g = \chi_{[0, 1]} \in L^1[0, 1]^* = L^\infty[0, 1]$, we have $\int_0^1 f_n g \, dx = 1$ for all n and does not converge to $\int_0^1 fg \, dx = 0$

Example

Consider the function γ defined in $\mathbb{R}$ by $\gamma(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ Define the sequence $\{f_n\}$ by $f_n(x) = \gamma(x - n)$



We have $f_n \to 0$ pointwise on $\mathbb{R}$, $f_n \in L^p(\mathbb{R})$ with $\|f_n\|_p = \left(\frac{2}{p+1}\right)^{1/p}$.
Let $A \subset \mathbb{R}$ be any measurable set with $m(A) < \infty$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $m(A \cap \{x : |x| \geq N\}) < \epsilon$. Then for $n > N$ we have

$$0 \leq \int_A f_n dx = \int_{A \cap \{x:\ |x| \geq N\}} 1 dx \leq \epsilon.$$

This implies $\lim_{n\to\infty} \int_A f_n dx = 0$ and so $f_n \rightharpoonup 0$ in $L^p(\mathbb{R})$ for $p > 1$.
However, $\{f_n\}$ does not converge weakly to 0 in $L^1(\mathbb{R})$. Indeed, for the function $g \equiv 1 \in L^\infty(\mathbb{R}) = L^1(\mathbb{R})^*$, we have $\int_{\mathbb{R}} f_n g dx = 1$ for all n and $\int_{\mathbb{R}} 0 g dx = 0$.

Theorem (2)

Let $E \subset \mathbb{R}^n$ be measurable and $1 < p < \infty$. If $\{f_n\} \subset L^p(E)$ is a bounded and converges pointwise to f a.e. on E, then $f_n \rightharpoonup f$ in $L^p(E)$.

Proof.

We first show that $f \in L^p(E)$. Let $M > 0$ be such that $\|f_n\|_p \leq M$ for all n. Since $|f_n|^p \in L^1(E)$, then it follows from Fatou's lemma that

$$\int_E |f|^p \, dm \leq \liminf_{n\to\infty} \int_E |f_n|^p \, dm \leq M^p \text{ and so } f \in L^p(E).$$

To check that $f_n \rightharpoonup f$, it is enough to verify that (Theorem 1) for every $A \subset E$ with $m(E) < \infty$, we have $\lim_{n\to\infty} \int_A f_n dm = \int_A f dm.$

We know (Lecture 26) that since the sequence $\{f_n\}$ is bounded in $L^p(E)$, then it is uniformly integrable. As a consequence, the Vitali Convergence Theorem implies $\lim_{n\to\infty} \int_A f_n dm = \int_A f dm.$ □

Theorem (3)

Let $E \subset \mathbb{R}^n$ be measurable, $1 < p < \infty$, and $\{f_n\} \subset L^p(E)$. Suppose theat $f_n \rightharpoonup f$ in L^p, then

$$f_n \to f \text{ in } L^p(E) \text{ if and only if } \lim_{n\to\infty} \|f_n\|_p = \|f\|_p \, .$$

Proof.

"$\Longrightarrow$" We already know that if a sequence converges strongly in a Banach space, then it converges weakly and the sequence of the norms converge to the norm of the limit.

"$\Longleftarrow$" Suppose that $f_n \rightharpoonup f$ in L^p and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. We are going to give the proof when $p \geq 2$ and use the following inequality (to be proved in Lemma 2))

$$\exists c > 0 \text{ such that } c|t|^p \leq |1+t|^p - 1 - pt \quad \forall t \in \mathbb{R}.$$

Use this inequality with $t = \dfrac{f_n - f}{f}$ and multiply by $|f|^p$ to get $c|f_n - f|^p \leq |f_n|^p - |f|^p - p\,\mathrm{sgn}(f)|f|^{p-1}(f_n - f)$.

Notice that for $g = \mathrm{sgn}(f)|f|^{p-1}$, after integration and passage to the limit we get

$$c\,\|f_n - f\|_p^p \leq \|f_n\|_p^p - \|f\|_p^p - p\int_E g(f_n - f)dm \longrightarrow 0$$

Lemma (2)

Let $p \geq 2$ then there exists $c > 0$ such that $c|t|^p \leq |1+t|^p - 1 - pt$ for all $t \in \mathbb{R}$.

Proof.

Consider the function $f(t)$ defined for $t \neq 0$ by $f(t) = \dfrac{\alpha(t)}{|t|^p}$ with $\alpha(t) = |1+t|^p - 1 - pt$. Then $\lim_{t \to 0} f(t) = \infty$ and $\lim_{|t| \to \infty} f(t) = 1$. Since $\alpha(0) = 0$ and for $t > 0$ we have $\alpha'(t) = p((1+t)^{p-1} - 1) > 0$, then $\alpha(t) > 0$ and $f(t) > 0$. Therefore, f has a positive minimum value c_1 on $t > 0$. A similar argument shows that f has a positive minimum c_2 for $t < 0$. The estimate of the lemma follows for $c = \min(c_1, c_2)$ □

A consequence of the previous theorem and Fatou's Lemma is the following.

Corollary (2)

Let $E \subset \mathbb{R}^n$ be measurable, $1 < p < \infty$, and $\{f_n\} \subset L^p(E)$. Suppose theat $f_n \rightharpoonup f$ in L^p, then $\{f_n\}$ has a subsequence that converges strongly to f if and only if $\|f\|_p = \liminf_{n \to \infty} \|f_n\|_p$.

Theorem 3 does not extend to the case $p = 1$ as the following example shows.

Example

Let $f_n(x) = 1 + \sin(nx)$. It follows from the previous example that $f_n \rightharpoonup f = 1$ in $L^1[-\pi, \pi]$. However, f_n does not converge strongly to f (see previous example). But since each f_n is nonnegative, then (as a consequence of the weak convergence in L^1) we also have $\|f\|_1 = \lim_{n \to \infty} \|f_n\|_1$.