Real Analysis MAA 6616
Lecture 28
Weak Sequential Compactness

Examples of the previous lecture show that there exist bounded sequences in L^p that fail to have any strongly convergent subsequence in L^p. However, we will see in this lecture that the situation is different for weak convergence. First we need the following theorem.

Theorem (1)

Let $(X, \|\cdot\|)$ be a separable normed spaces and let $\{T_n\}_n$ be a bounded sequence in the dual space X^*: That is, there exists $M > 0$ such that $|T_n f| \leq M \|f\|$ for all $f \in X$ and $n \in \mathbb{N}$. Then there is a subsequence $\{T_{n_j}\}_j$ of $\{T_n\}_n$ that converges to $T \in X^*$: $\lim_{j\to\infty} T_{n_j} f = Tf$ for all $f \in X$.

Proof.

Since X is separable, it has a dense countable subset $\{f_n\}_n$. Since $|T_n f_1| \leq M \|f_1\|$ for all n, then the sequence of real numbers $\{T_n f_1\}$ has convergent subsequence: There is an increasing sequence of integers $\mu_{1,j}$ and $\alpha_1 \in \mathbb{R}$ such that $\lim_{j\to\infty} T_{\mu_{1,j}} f_1 = \alpha_1$.

Repeat this argument with $\{T_{\mu_{1,j}}\}_j$ replacing $\{T_n\}_n$ and f_2 replacing f_1. We obtain then an increasing subsequence $\mu_{2,j}$ of $\mu_{1,j}$ and $\alpha_2 \in \mathbb{R}$ such that $\lim_{j\to\infty} T_{\mu_{2,j}} f_i = \alpha_i$ for $i = 1, 2$. Continue this process inductively to obtain an increasing sequence $\mu_{r,j}$ of $\mu_{r-1,j}$ and a real number $\alpha_r \in \mathbb{R}$ such that $\lim_{j\to\infty} T_{\mu_{r,j}} f_k = \alpha_k$ for $k = 1, \cdots, r$.

Now for $m \in \mathbb{N}$, let $\sigma_m = \mu_{m,m}$. Consider the subsequence $\{T_{\sigma_m}\}_m \subset \{T_n\}_n$. For any $k \in \mathbb{N}$, we have $\lim_{m\to\infty} T_{\sigma_m} f_k = \alpha_k$. Next we prove that $\{T_{\sigma_m} f\}_m$ is Cauchy (in $\mathbb{R}$) for every $f \in X$. Let $f \in X$ and $\epsilon > 0$. Since $\{f_k\}_k$ is dense in X, let f_k such that $\|f - f_k\| < \frac{\epsilon}{4M}$. The sequence $\{T_{\sigma_m} f_k\}_m$ is Cauchy and so there exists N such that for every $n, m > N$ we have $|T_{\sigma_m} f_k - T_{\sigma_n} f_k| < \frac{\epsilon}{2}$. We have

$$
\begin{aligned}
|T_{\sigma_m} f - T_{\sigma_n} f| &\leq |T_{\sigma_m} f - T_{\sigma_m} f_k| + |T_{\sigma_m} f_k - T_{\sigma_n} f_k| + |T_{\sigma_n} f_k - T_{\sigma_n} f| \\
&\leq M \|f - f_k\| + \frac{\epsilon}{2} + M \|f - f_k\| \leq \epsilon
\end{aligned}
$$

□

Theorem (2)

Let $E \subset \mathbb{R}^n$ be measurable and $1 < p < \infty$. Then every bounded sequence in $L^p(E)$ has a subsequence that converges **weakly** *in $L^p(E)$.*

Proof.

Let $\{f_n\}_{n=1}^{\infty} \subset L^p(E)$ be a bounded sequence. Hence there is $M > 0$ such that $\|f_n\|_p \leq M$ for all n. Let q be the conjugate of p ($p^{-1} + q^{-1} = 1$). For every n consider the functional $T_n \in L^q(E)^* \cong L^p(E)$ be defined by

$$T_n(g) = \int_E f_n g dx \text{ for all } g \in L^q(E).$$

Then $|T_n g| \leq \|f_n\|_p \|g\|_q \leq M \|g\|_q$. Hence the sequence $\{T_n\}_n$ is bounded in $L^q(E)^*$. Theorem 1 implies there is a subsequence $\{T_{n_j}\}_j$ and $T \in L^q(E)^*$ such that $\lim_{j\to\infty} T_{n_j} g = Tg$ for all $g \in L^q(E)$. The Riesz Representation Theorem implies that there exists a unique $f \in L^p(E)$ such that $Tg = \int_E fgdx$ for all $g \in L^q(E)$. This means

$$\lim_{j\to\infty} \int_E f_{n_j} g dx = \int_E fgdx \text{ for all } g \in L^q(E).$$

This is equivalent to $f_{n_j} \rightharpoonup f$ in $L^p(E)$. □

Remark (1)

The above theorem does not extend to the case $p = 1$ as the following example shows.

In $L^1[0, 1]$ consider the sequence $\{f_n\}_n$ given by $f_n = n\chi_{[0, 1/n]}$. This sequence is bounded in $L^1[0, 1]$ since $\|f_n\|_1 = \int_0^1 n\chi_{[0, 1/n]} = 1$. Now we show by contradiction that $\{f_n\}$ has no subsequence that converges weakly in $L^1[0, 1]$. Suppose that $\{f_{n_j}\}_j$ converges weakly to $f \in L^1[0, 1]$. Then for $0 < c < d \leq 1$, we would have

$$\int_0^1 f\chi_{[c, d]} dx = \lim_{j\to\infty} \int_0^1 f_{n_j}\chi_{[c, d]} dx = 0.$$

Since c, d are arbitrary in $(0, 1]$, then $f = 0$ a.e. in $[0, 1]$. This implies

$$0 = \int_0^1 fdx = \lim_{j\to\infty} \int_0^1 f_{n_j} dx = 1$$

A contradiction.

Let $(X, \|\cdot\|)$ be a normed space. A subset $K \subset X$ is said to be weakly sequentially compact if every sequence $\{f_n\}_n \subset K$ has a subsequence that converges weakly to an element in K.

Theorem (3)

Let $E \subset \mathbb{R}^n$ be measurable and $1 < p < \infty$. Then the closed unit ball in $L^p(E)$ is weakly sequentially compact. The closed unit ball in $L^p(E)$ is the set

$$\mathcal{B}^p(E) = \{f \in L^p(E) : \|f\|_p \leq 1\}.$$

Proof.

Let $\{f_n\}_n \subset \mathcal{B}^p(E)$. Then $\|f_n\|_p \leq 1$ for all n, and it follows from Theorem 2 that there exists $f \in L^p(E)$ and a subsequence $\{f_{n_j}\}_j$ such that $f_{n_j} \rightharpoonup f$. It remains to very that $f \in \mathcal{B}^p(E)$. This follows from

$\|f\|_p \leq \liminf_{j \to \infty} \left\|f_{n_j}\right\|_p \leq 1.$ □

Theorem (4)

Let $E \subset \mathbb{R}^n$ be measurable and $1 < p < \infty$. Suppose that $\{f_n\}_n \subset L^p(E)$ converges weakly to $f \in L^p(E)$. Then there exists a subsequence $\{f_{n_j}\}_j$ such that its sequence of arithmetic means converges strongly to f. That is if

$$g_m = \frac{f_{n_1} + f_{n_2} + \cdots + f_{n_m}}{m} \quad \text{then} \quad g_m \to f \ \text{ in } \ L^p(E)$$

Proof.

Case $p = 2$. We can assume $f = 0$ (after replacing f_n by $f_n - f$). The weak converhgence of $\{f_n\}$ implies that the sequence is bounded. Let $M > 0$ be such that $\|f_n\|_2^2 \le M$ for all n.

Let $n_1 = 1$. Since $f_n \rightharpoonup 0$ in L^2 and $f_{n_1} \in L^2(E) \cong L^2(E)^*$, then there exists $n_2 > n_1$ such that $\left|\int_E f_{n_1} f_{n_2}\, dx < 1\right|$.

Suppose that we have $n_1 < n_2 < \cdots < n_k$ and functions $f_{n_1}, \cdots, f_{n_k}$ such that $\int_E (f_{n_1}, \cdots, f_{n_j})^2 dx \le (2 + M)j$ for $j = 1, \cdots, k$. Then since $f_n \rightharpoonup 0$ in L^2 and $f_{n_1} + \cdots + f_{n_k} \in L^2(E)$, then there exists $n_{k+1} > n_k$ such that $\int_E (f_{n_1} + \cdots + f_{n_k}) f_{n_{k+1}}\, dx \le 1$. It follows that

$$\begin{aligned}\int_E (f_{n_1} + \cdots + f_{n_{k+1}})^2 dx &= \int_E (f_{n_1} + \cdots + f_{n_k})^2 dx + 2\int_E (f_{n_1} + \cdots + f_{n_k}) f_{n_{k+1}}\, dx + \int_E f_{n_{k+1}}^2\, dx \\ &\le (2 + M)k + 2 + M = (2 + M)(k + 1)\end{aligned}$$

Let $\{g_m\}_m$ be the arithmetic mean of the constructed subsequence $\{f_{n_j}\}_j$. Then

$$\|g_m\|_2^2 = \int_E \left(\frac{f_{n_1} + \cdots + f_{n_m}}{m}\right)^2 dx \le \frac{2 + M}{m} \longrightarrow 0$$

□

A set $C \subset X$ is said to be convex if for every $f, g \in C$ and $\lambda \in [0,\ 1]$, $\lambda f + (1-\lambda)g \in C$. The set C is said to be closed if for every sequence $\{f_n\}_n \subset C$ such that $f_n \longrightarrow f$ in X, then $f \in C$.

A mapping $T : C \subset X \longrightarrow \mathbb{R}$ is said to be continuous on C if for every sequence $\{f_n\}_n \subset C$ such that $f_n \to f \in C$, we have $Tf_n \to Tf \in \mathbb{R}$. Note that when T is linear, the continuity of T is equivalent to T bounded.
When C is convex, the operator T is said to be convex if for every $f, g \in C$ and $\lambda \in [0,\ 1]$ we have $T(\lambda f + (1-\lambda)g) \le \lambda Tf + (1-\lambda)Tg$.

Examples

1. Let $E \subset \mathbb{R}^n$ be measurable, $1 \le p < \infty$, and $g \in L^p(E)$ with g nonnegative. The set $C = \{f \text{ measurable on } E : \ |f| \le g \text{ a.e.}\}$ is a closed and convex subset of $L^p(E)$. Indeed, $C \subset L^p(E)$ follows from $|f| < g \implies \|f\|_p \le \|g\|_p$. For $f, h \in C$ and $\lambda \in [0,\ 1]$ we have $|\lambda f + (1-\lambda)h| \le \lambda|f| + (1-\lambda)|h| \le g$. If $\{f_n\} \subset C$ is such that $f_n \to f \in L^p(E)$, then there exists a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \to f$ pointwise a.e. Moreover $\left|f_{n_k}\right| \le g$ for all k implies $|f| \le g$ and so $f \in C$.
2. Let $E \subset \mathbb{R}^n$ be measurable and $1 \le p < \infty$. The ball $B\{f \in L^p(E) : \ \|f\|_p \le 1\}$ is closed and convex in $L^p(E)$. That B is closed follows rom the triangle inequality. Indeed if $\{f_n\}_n \subset C$ converges to $f \in L^p(E)$. Then $\|f_n - f\|_p \to 0$ and
$$\|f\|_p \le \|f_n\|_p + \|f_n - f\|_p \le 1 + \|f_n - f\|_p \implies \|f\|_p \le 1$$
3. Let $E \subset \mathbb{R}^n$ be measurable with finite measure, $1 \le p < \infty$, and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, convex, and such that there exist $a, b \ge 0$ such that $|\phi(t)| \le a + b|t|^p$ for all $t \in \mathbb{R}$.
Consider the operator $T : L^p(E) \longrightarrow \mathbb{R}$ given by $T(f) = \displaystyle\int_E \phi \circ f \, dx$. Then T is continuous and convex (the continuity will be shown later). The convexity follows from that of ϕ

Lemma (1)

Let $E \subset \mathbb{R}^n$, $1 < p < \infty$, and $C \subset L^p(E)$ be closed, bounded, and convex. Let $T : C \longrightarrow \mathbb{R}$ be continuous and convex. If $\{f_n\}_n \subset C$ and $f_n \rightharpoonup f$ in L^p, then $f \in C$ and $T(f) \leq \liminf_{n\to\infty} T(f_n)$.

Proof.

It follows from Banach-Saks Theorem that there exists a subsequence $\{f_{n_j}\}_j$ such that the sequence $\{\mu(f_{n_j})\}_j$ of arithmetic means converges to f in L^p. Since $\{\mu(f_{n_j})\}_j \subset C$ and C closed, then $f \in C$.

Let $\alpha = \liminf_{n\to\infty} T(f_n)$. There exists a subsequence $\{f_{n_k}\}_k$ such that $\alpha = \lim_{k\to\infty} T(f_{n_k})$. We can assume that the arithmetic mean of $\{f_{n_k}\}_k$ converges to f in L^p. Note that if a sequence of real numbers x_n converges to l, then its sequence of arithmetic means $\mu(x_n)$ also converges to l. It follows from the continuity of T and its convexity $T(\mu(g_j)) \leq \mu(T(g_j))$ that

$$T(f) = \lim_{k\to\infty} T(\mu(f_{n_k})) \leq \lim_{k\to\infty} \mu T(f_{n_k}) = \lim_{k\to\infty} T(f_{n_k}) = \alpha = \liminf_{n\to\infty} T(f_n)$$

□

Theorem (5)

Let $E \subset \mathbb{R}^n$, $1 < p < \infty$, and $C \subset L^p(E)$ be closed, bounded, and convex. Let $T : C \longrightarrow \mathbb{R}$ be continuous and convex. Then T attains a minimum value in C. That is there exists $f_0 \in C$ such that

$$T(f_0) \leq T(f) \quad \text{for all } f \in C.$$

Proof.

First we prove that $T(C)$ is bounded below (i.e. there exists $A \in \mathbb{R}$ such that $T(f) \geq A$ for all $f \in C$). By contradiction, if $T(C)$ were not bounded below, then there would be a sequence $\{f_n\} \subset C$ such that $T(f_n) \to -\infty$. Since $C \subset L^p(E)$ is bounded, then so is the sequence. We can therefore (by taking a a subsequence if necessary) assume that $f_n \rightharpoonup f$ in L^p. Lemma 1 implies that $f \in C$ and $T(f) \leq \liminf_{n\to\infty} T(f_n) = -\infty$. A contradiction and so $T(C)$ is bounded below.

Let $m = \inf\{T(f) : f \in C\}$. There exists a sequence $\{f_n\}_n \subset C$ such that $T(f_n) \to c$. We can assume that $f_n \rightharpoonup f_0$ in L^p. Lemma 1 implies that $f_0 \in C$ and $T(f_0) \leq \liminf_{n\to\infty} T(f_n) = m$. Therefore $T(f_0) = m$. □

Corollary (1)

Let $E \subset \mathbb{R}^n$ with $m(E) < \infty$, $1 < p < \infty$, and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, convex, and such that there exist $a, b \geq 0$ such that

$$|\phi(t)| \leq a + b|t|^p \quad \text{for all} \quad t \in \mathbb{R}.$$

Then there exists $f_0 \in L^p(E)$ with $\|f_0\|_p \leq 1$ such that

$$\int_E \phi \circ f_0 dx = \min\{\int_E \phi \circ f\, dx : \; f \in L^p(E), \; \|f\|_p \leq 1\}.$$