Real Analysis MAA 6616
Lecture 29
Abstract Measures
Measurable and Integrable Functions
Signed Measures

Abstract Measure

Let X be a set $\mathcal{A}$ be a σ-algebra of subsets of X. That is $\mathcal{A}$ satisfies:

- $X \in \mathcal{A}$;
- If $A \in \mathcal{A}$, then its complement $A^c = X \backslash A$ is also in $\mathcal{A}$ ($\mathcal{A}$ closed under complement); and
- if $A_1, A_2, \cdots$ are in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ ($\mathcal{A}$ closed under countable union).

These conditions imply that $\emptyset \in \mathcal{A}$, $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$, $\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$ and

$\limsup A_i = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$ are in $\mathcal{A}$, and $A \backslash B \in \mathcal{A}$ for $A, B \in \mathcal{A}$.

A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \longrightarrow [0, \ \infty]$ such that

- $\mu(\emptyset) = 0$; and
- when $\{A_j\}_{j=1}^{\infty} \in \mathcal{A}$ is a collection of disjoint sets, then $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$.

The triplet $(X, \mathcal{A}, \mu)$ is called a measure space and the element of $\mathcal{A}$ are measurable sets.

Examples

Let X be any set, $\mathcal{A} = 2^X$ (the set of all subsets of X), and for $A \in \mathcal{A}$, $\mu(A)$ the number of elements of A, if A finite, otherwise $\mu(A) = \infty$: μ is the counting measure.
Let $X = \mathbb{R}^n$, $\mathcal{A}$ the σ-algebra of Lebesgue measurable sets, and $\mu = m$ is the Lebesgue measure.
Let $x_0 \in X$, $\mathcal{A}$ any σ-algebra, and the Dirac measure at x_0 given by $\delta_{x_0} : \mathcal{A} \longrightarrow \mathbb{R}$ given by $\delta_{x_0}(A) = 1$ if $x_0 \in A$ and $\delta_{x_0}(A) = 0$ if $x_0 \notin A$. $(X, \mathcal{A}, \delta_{x_0})$ is the Dirac measure space.

The arguments used for the Lebesgue measure can be used to prove the following results. The details are left as exercises.

Proposition (1)

A measure μ satisfies the following;

1. *If $A, B \in \mathcal{A}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.*
2. *If $\{A_i\}_i \subset \mathcal{A}$, then $\mu\left(\bigcup_i A_i\right) \leq \sum_i \mu(A_i)$.*
3. *If $\{A_i\}_i \subset \mathcal{A}$ is an ascending sequence, then $\mu\left(\bigcup_i A_i\right) = \lim_{i\to\infty} \mu(A_i)$.*
4. *If $\{A_i\}_i \subset \mathcal{A}$ is an descending sequence and one of the A_i's has finite measure, then $\mu\left(\bigcap_i A_i\right) = \lim_{i\to\infty} \mu(A_i)$.*

Note that in (4), if none of the sets has finite measure, then the conclusion might not hold as the following example shows: Let $X = \mathbb{N}$ and μ be the counting measure. Let $A_j = \{x \in \mathbb{N} : \ x \geq i\}$. Then $A_i \searrow$, $\mu(A_i) = \infty$ for all i, and we have $\bigcap_i A_i = \emptyset$ and so $0 = \mu(\bigcap_i A_i) \neq \lim_{i\to\infty} \mu(A_i)$
Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\mu(X) < \infty$, μ is said to be a finite measure and $(X, \mathcal{A}, \mu)$ a finite measure space. If $\mu(X) = \infty$ and if there exists $\{A_j\}_j \subset \mathcal{A}$ such that $X = \bigcup_j A_j$ and $\mu(A_j) < \infty$ for all j, the μ is said to a σ-finite measure and $(X, \mathcal{A}, \mu)$ a σ-finite measure space. A set $N \subset X$ (not necessarily in $\mathcal{A}$) is said to be a null set if there exists $Z \in \mathcal{A}$ such that $N \subset Z$ and $\mu(Z) = 0$. The measure space $(X, \mathcal{A}, \mu)$ is said to be complete if every null set is contained in $\mathcal{A}$. Given a measure space $(X, \mathcal{A}, \mu)$, there exists a smallest σ-algebra $\overline{\mathcal{A}}$ containing $\mathcal{A}$ and all null sets and a measure $\overline{\mu} : \overline{\mathcal{A}} \longrightarrow [0, \ \infty]$ such that $(X, \overline{\mathcal{A}}, \overline{\mu})$ is complete. It is the completion of $(X, \mathcal{A}, \mu)$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$ and $f : E \longrightarrow \overline{\mathbb{R}}$. f is said to be $\mathcal{A}$-measurable or simply measurable if for every $a \in \mathbb{R}$ the set $\{f > a\} = \{x \in E : f(x) > a\}$ is $\mathcal{A}$-measurable (i.e. $\{f > a\} \in \mathcal{A}$).

A property $(\mathcal{P})$ is said to hold almost everywhere in a set $E \in \mathcal{A}$ if there exists a set $Z \subset E$ such that $\mu(Z) = 0$ and $(\mathcal{P})$ holds in $E \backslash Z$.

Let $E \subset X$. The characteristic function of E is: $\chi_E : X \longrightarrow \mathbb{R}$ given by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. A simple function ϕ on X is a finite linear combination of characteristic function: $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ for some disjoint sets $E_1, \cdots, E_n \in X$ and $a_1, \cdots, a_n \in \mathbb{R}$.

Proposition (2)

Let $f : E \longrightarrow \overline{\mathbb{R}}$. The following conditions are equivalent:

- $\{f > a\} \in \mathcal{A}$ *for all* $a \in \mathbb{R}$;
- $\{f \geq a\} \in \mathcal{A}$ *for all* $a \in \mathbb{R}$;
- $\{f < a\} \in \mathcal{A}$ *for all* $a \in \mathbb{R}$;
- $\{f \leq a\} \in \mathcal{A}$ *for all* $a \in \mathbb{R}$.

In addition if any of the above conditions hold, then $\{f = a\} \in \mathcal{A}$ *for all* $a \in \mathcal{A}$

Proposition (3)

If X is a metric space and $\mathcal{A}$ contains all open sets of X, then a continuous function $f : E \longrightarrow \mathbb{R}$ is measurable.

Theorem (1)

1. *If* $f, g : E \longrightarrow \overline{\mathbb{R}}$ *be measurable, then so are the functions* $f + g$, cf *(with c a real number),* fg, $\max(f, g)$, $\min(f, g)$, $1/f$ *(provided* $f \neq 0$ *on E).*
2. *If* $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous and f measurable over E, then* $\psi \circ f$ *is measurable. In particular, the functions,* $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$, *and* $|f|^p$ *with* $p > 0$ *are measurable.*
3. *A simple function* $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ *is measurable if and only if each set* E_j *is measurable.*
4. *If* $\{f_j\}_{j=1}^{\infty}$ *is a sequence of measurable functions on a set E, then so are the functions* $\sup_j f_j$, $\inf_j f_j$, $\limsup_{j\to\infty} f_j$, $\liminf_{j\to\infty} f_j$, *and* $\lim_{j\to\infty} f_j$ *if it exists.*
5. *If* $f : E \longrightarrow \overline{\mathbb{R}}$ *is nonnegative and measurable, then there exists a sequence of measurable, nonnegative, simple functions* $\{\phi_j\}_j$ *such that* $\phi_j \nearrow f$.

We have an analogue version of Egorov's Theorem

Theorem (2)

Let $(X, \mathcal{A}, \mu)$ *be a measure space and* $E \in \mathcal{A}$ *with* $\mu(E) < \infty$. *Let* $\{f_j\}_j$ *be a sequence of measurable functions on E such that* f_j *is finite a.e. and* f_j *converges a.e. to a function f that is finite a.e. Then for any given* $\epsilon > 0$, *there exists a set* $A \in \mathcal{A}$ *with* $A \subset E$ *and* $\mu(E \backslash A) < \epsilon$, *such that* $\{f_j\}_j$ *converges uniformly to f on the set A.*

μ-Integrable Functions

If $\phi = \sum_j a_j \chi_{E_j}$ is a nonnegative simple function on $E \in \mathcal{A}$, we define the Lebesgue μ-integral or simple the integral of ϕ as $\displaystyle\int_E \phi d\mu = \sum_j a_j \mu(E_j)$. Where we use the convention that if $a_j = 0$ and $\mu(E_j) = \infty$, then $a_j \mu(E_j) = 0$. Note that the integral of ϕ as defined could be ∞.

Let $f : E \longrightarrow \overline{\mathbb{R}}$ be a nonnegative measurable function. Set

$$\int_E f d\mu = \sup \left\{ \int_E \phi d\mu : \ 0 \leq \phi \leq f \, , \ \ \phi \text{ simple function} \right\}$$

The function f is said to be integrable if $\displaystyle\int_E f d\mu < \infty$.

Given a measurable function $f : E \longrightarrow \overline{\mathbb{R}}$, if $|f|$ is integrable over E, then so are f^+ and f^-. In this case the function f is said to be integrable over E and define its integral over E as

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$

It follows from the triangle inequality that

$$\left| \int_E f d\mu \right| \leq \int_E |f| d\mu$$

For $E \in \mathcal{A}$, denote by $\mathcal{L}(E, d\mu)$ the space of functions that are μ-integrable over E. It can be shown that $\mathcal{L}(E, d\mu)$ is a linear space.

Some properties of μ-integrable functions

- If $f \geq 0$ and $\int_E f d\mu = 0$, then $f = 0$ a.e.
- If $f \in \mathcal{L}(E, d\mu)$ and $\int_A f d\mu = 0$ for every $A \in \mathcal{A}$ with $A \subset E$, then $f = 0$ a.e.
- Fatou's Lemma: For $\{f_n\}_n \subset \mathcal{L}(E, d\mu)$ and $f_n \geq 0$, then
$$\int_E \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu$$
- Monotone Convergence Theorem: Suppose $\{f_n\}_n \subset \mathcal{L}(E, d\mu)$, $f_n \geq 0$, and $f_n \nearrow f$ a.e., then
$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu$$
- Dominated Convergence Theorem: Let $\{f_n\}_n$ is a sequence of μ-measurable functions over $E \in \mathcal{A}$ and $f_n \to f$ pointwise a.e. Suppose that there exists $g \in \mathcal{L}(E, d\mu)$ such that $|f_n| \leq g$ for all n, then
$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

A signed measure over X is a function $\mu : \mathcal{A} \longrightarrow (-\infty, \infty]$ defined over a σ-algebra $\mathcal{A}$ over X such that $\mu(\emptyset) = 0$ and μ is countably additive in the sense that if $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$ is disjoint, then $\mu(\bigcup_j A_j) = \sum_j \mu(A_j)$, where the series $\sum_j \mu(A_j)$ is absolutely convergent ($\sum_j |\mu(A_j)| < \infty$) whenever $\mu(\bigcup_j A_j) < \infty$. The measure μ is said to be positive if it does not take negative values.

A set $P \in \mathcal{A}$ (resp. $N \in \mathcal{A}$) is said to be μ-positive (resp. μ-negative) if $\mu(E) \geq 0$ (resp. $\mu(E) \leq 0$) whenever $E \in \mathcal{A}$ with $E \subset P$ (resp. $E \subset N$). A set $S \in \mathcal{A}$ is said to be a μ-null if $\mu(E) = 0$ for every $E \in \mathcal{A}$ with $E \subset S$.

Example

Let $f \in \mathcal{L}(\mathbb{R}^n, dm)$ (m is the Lebesgue measure) and let M be the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$. Define a signed measure μ on $\mathcal{M}$ by

$$\mu(E) = \int_E f dm \, .$$

Note that if $\{E_j\}_j \subset \mathcal{M}$ is disjoint then f (and so $|f|$) is integrable over $\bigcup_j E_j$. We have $\mu(\bigcup_j E_j) = \int_{\bigcup_j E_j} f dm < \infty$.

Also

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} f dm \right| \leq \sum_j \int_{E_j} |f| dm = \int_{\bigcup_j E_j} |f| dm < \infty .$$

The series $\sum_j \mu(E_j)$ converges absolutely and $\mu(\bigcup_j E_j) = \sum_j \mu(E_j)$.

Let $P = \{x \in \mathbb{R}^n : f(x) \geq 0\}$. If $E \subset P$ is measurable, then $f \geq 0$ on E and $\mu(E) = \int_E f dm \geq 0$. Hence P is a positive set for μ. Similarly the set $N = \{x \in \mathbb{R}^n : f(x) < 0\}$ is a negative set for μ.

Proposition (2)

Let $\mu : \mathcal{A} \longrightarrow (-\infty, \infty]$ be a signed measure. If $A \in \mathcal{A}$ is such that $\mu(A) < 0$, then there exists a negative set $N \subset A$ with $\mu(N) < 0$.

Proof.

If A is a negative set, take $N = A$, and we are done. If A is not a negative set, then it has a subset with positive measure. Let n_1 be the smallest positive integer such that there exists a set $E_1 \subset A$ with $\mu(E_1) > 1/n_1$. Let $B_1 = A \backslash E_1$. Then $B_1 \subset A$ with $\mu(B_1) = \mu(A) - \mu(E_1) < \mu(A) < 0$. If B_1 is a negative set, take $N = B_1$ and we are done. If not, then B_1 contains a subset with a positive measure. Let $n_2 > n_1$ be the smallest integer such that there exist a set $E_2 \subset B_1$ with $\mu(E_2) > 1/n_2$. Then $E_1 \cap E_2 = \emptyset$. Let $B_2 = B_1 \backslash E_2 = A \backslash (E_1 \cup E_2)$. We have $\mu(B_2) = \mu(A) - (\mu(E_1) + \mu(E_2)) < 0$. If B_2 is a negative set, we are done. Other repeat the construction by induction. At the j-th step, we will have increasing sequence of integers $n_1 < n_2 < \cdots < n_j$, disjoint sets $E_1, \cdots, E_j \in \mathcal{A}$ with $E_k \subset A$ and $\mu(E_k) > 1/n_k$ for $k = 1, \cdots, j$. Let $B_j = B_{j-1} \backslash E_j = A \backslash (\bigcup_{k=1}^{j} E_k)$. We have $\mu(B_j) < \mu(A) < 0$. If B_j is a negative set, we are done. If not we obtain an infinite sequence. In this case define $N = \bigcap_{j=1}^{\infty} B_j = A \backslash \bigcup_{j=1}^{\infty} E_j$. Then $N \subset A$ and
$\mu(N) = \mu(A) - \sum_j \mu(E_j) < \mu(A) < 0$. It remains to verify that N is a negative set. By contradiction, suppose that there exists a set $F \subset N$ with $\mu(F) > 0$. Then there would be a positive integer r such that $\mu(F) > 1/r$. This would mean that $F \subset E_j$ for $n_j > r$ and $F \not\subset B_j$ which is a contradiction. □