Real Analysis MAA 6616
Lecture 3
Sequences of Real Numbers

A sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \longrightarrow \mathbb{R}$. Usually $f(j)$ is denoted a_j and the sequence denoted $\{a_j\}$. The number a_j corresponding to the index j is called the j-th term of the sequence. The sequence is said to be bounded if there exists $M \geq 0$ such that $|a_j| \leq M$ for all j. A sequence $\{a_j\}$ is increasing (resp. decreasing) if $a_j \leq a_{j+1}$ (resp. $a_j \geq a_{j+1}$) for all j. The sequence is monotone if it is either increasing or decreasing.

A sequence $\{a_j\}$ is said to converge to a number c (called the limit) if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_j - c| \leq \epsilon \quad \forall j \geq N .$$

In this case we use the notation $c = \lim_{j \to \infty} a_j$ or $a_j \longrightarrow c$.

Proposition

Suppose that the sequence $\{a_j\}$ converges to c.

1. *The limit c is unique;*
2. *The sequence is bounded;*
3. *If $a_j \leq M$ for all j, then $|c| \leq M$*

Proof.

1. If c and c' are limits of the sequence $\{a_j\}$, then given $\epsilon > 0$, there are N and N' in $\mathbb{N}$ such that $|c - a_j| \leq \epsilon$ for all $j \geq N$ and $|c' - a_j| \leq \epsilon$ for all $j \geq N'$. Let $j > \max(N, N')$, we have

$$|c - c'| = |c - a_j + a_j - c'| \leq |c - a_j| + |a_j - c'| \leq 2\epsilon$$

   Since $\epsilon > 0$ is arbitrary, then $c = c'$.

2. Let $\epsilon = 1$. There exists $N \in \mathbb{N}$ such that $|c - a_j| \leq 1$ for all $j > N$. Hence for $j > N$ we have

$$|a_j| = |a_j - c + c| \leq |a_j - c| + |c| \leq 1 + |c| \ .$$

   Let $M = \max(|a_1|, \cdots, |a_N|, 1 + |c|)$. Then $|a_j| \leq M$ for all j.

3. Left as an exercise

□

Theorem

A monotone sequence is convergent if and only if it is bounded.

Proof.

We know from the previous proposition that a convergent sequence is bounded. To prove the theorem we need only show that a bounded monotone sequence is convergent. Suppose that $\{a_j\}$ is decreasing and bounded. Let $c = \inf_{j\in\mathbb{N}}\{a_j\}$. Let $\epsilon > 0$ be arbitrary. Since $c + \epsilon$ is not a lower bound of the sequence, then there exists $N \in \mathbb{N}$ such that $c \le a_N < c + \epsilon$. Since the sequence is decreasing, then $a_j \le a_N$ for all $j \ge N$. We have therefore $c \le a_j \le a_N < c + \epsilon$ for all $j \ge N$. That is $|a_j - c| < \epsilon$ for all $j \ge N$. Hence $a_j \longrightarrow c$. □

Let $\{a_n\}_n$ be a sequence in $\mathbb{R}$ and let $\{n_k\}_k$ be a a strictly increasing sequence in $\mathbb{N}$. The sequence $\{a_{n_k}\}_k$ is called a subsequence of $\{a_n\}$. The j-th term of $\{a_{n_k}\}_k$ is a_{n_j}.

Theorem

A bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof.

Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$ and let $M \geq 0$ such that $|a_n| \leq M$ for all n. For each $j \in \mathbb{N}$, consider closed set $E_j = \overline{\{a_k : k > j\}}$. Then $E_j \subset [-M, M]$ and $E_{j+1} \subset E_j$ for all j. It follows from the Nested Set Theorem that $\bigcap_{j=1}^{\infty} E_j \neq \emptyset$. Let $c \in \bigcap_{j=1}^{\infty} E_j$.

Since $c \in E_1 = \overline{\{a_k : k > 1\}}$, then we can find $k_1 \in \mathbb{N}$ such that $\left|c - a_{k_1}\right| < 1$. Similarly $c \in E_{k_1} = \overline{\{a_k : k > k_1\}}$, then there exists $k_2 > k_1$ such that $\left|c - a_{k_2}\right| < 1/2$. By induction, suppose that we have a $k_1 < k_2 < \cdots < k_j$ such that $\left|c - a_{k_j}\right| < 1/j$. Using the fact that $c \in E_{k_j} = \overline{\{a_k : k > k_j\}}$, we can find $k_{j+1} > k_j$ such that $\left|c - a_{k_{j+1}}\right| < 1/(j+1)$. Therefore the subsequence $\{a_{k_j}\}$ converges to c. □

A sequence $\{a_j\}$ in $\mathbb{R}$ is said to be Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m > N$.

Theorem

A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof.

Let $\{a_n\}$ be a sequence in $\mathbb{R}$.
"$\Longrightarrow$" Suppose that $\{a_n\}$ converges to c. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - c| < \epsilon/2$ for all $n > N$. For $m, n > N$, we have

$$|a_m - a_n| = |a_m - c + c - a_n| \leq |a_m - c| + |c - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that $\{a_n\}$ is Cauchy.
"$\Longleftarrow$" Suppose that $\{a_n\}$ is Cauchy. First we prove that the sequence is bounded. Let $\epsilon = 1$, then there exists $N_1 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ whenever $n, m \geq N_1$. Let $M = 1 + \max(|a_1|, \cdots, \left|a_{N_1}\right|)$. Then for $j \leq N_1$, $\left|a_j\right| < M$ and for $j > N_1$, we have

$$\left|a_j\right| = \left|a_j - a_{N_1} + a_{N_1}\right| \leq \left|a_j - a_{N_1}\right| + \left|a_{N_1}\right| < 1 + \left|a_{N_1}\right| \leq M$$

Hence the sequence is bounded. It follows then from the Bolzano-Weierstrass Theorem that $\{a_n\}$ has a convergent subsequence $\{a_{n_j}\}$. Let $c = \lim a_{n_j}$. We claim that the original sequence also converges to c. Indeed, let $\epsilon > 0$. Using the hypothesis that $\{a_n\}$ is Cauchy and that the subsequence converges, there is $N \in \mathbb{N}$ and $J \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$ for $n, m > N$ and $\left|a_{n_j} - c\right| < \epsilon/2$ for $j > J$. For $n > N$, let $m = n_j > N$ with $j > J$. Then

$$|a_n - c| = \left|a_n - a_{n_j} + a_{n_j} - c\right| \leq \left|a_n - a_{n_j}\right| + \left|a_{n_j} - c\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The proofs of the following properties of convergent sequences are left as exercises. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences.

- For $\lambda, \mu \in \mathbb{R}$, we have $\lim_{n\to\infty}(\lambda a_n + \mu b_n) = \lambda \lim_{n\to\infty} a_n + \mu \lim_{n\to\infty} b_n$.
- If there exists $p \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq p$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

A sequence $\{a_n\}$ is said to converge to infinity and write $\lim_{n\to\infty} a_n = \infty$ if for every $A > 0$, there exists $N \in \mathbb{N}$ such that $a_n > A$ for all $n > N$.
A sequence $\{a_n\}$ is said to converge to minus infinity and write $\lim_{n\to\infty} a_n = -\infty$ if for every $B < 0$, there exists $N \in \mathbb{N}$ such that $a_n < B$ for all $n > N$.

The limit superior and limit inferior of a sequence $\{a_n\}$ are defined by

$$\limsup a_n = \lim_{n\to\infty}[\sup\{a_k : k \geq n\}] \text{ and } \liminf a_n = \lim_{n\to\infty}[\inf\{a_k : k \geq n\}] .$$

Proposition

1. $\limsup a_n = s$ *if and only if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $a_n < s + \epsilon$ *for all* $n > N$ *and for every* $k \in \mathbb{N}$ *there exists* $n_k > k$ *such that* $s - \epsilon < a_{n_k}$ *(there are only finitely many* a_n*'s that are* $> s + \epsilon$ *and infinitely many that are* $> s - \epsilon$*)*
2. $\limsup a_n = \infty$ *if and only if* $\{a_n\}$ *is not bounded above.*
3. $\limsup a_n = -\liminf(-a_n)$.
4. $\{a_n\}$ *converges in* $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ *if and only if* $\limsup a_n = \liminf a_n$.

Proof.

1. "$\Longrightarrow$" Suppose $\limsup a_n = s$. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\sup\{a_k : k \geq n\} - s| < \epsilon$ for all $n > N$. Hence $a_k < s + \epsilon$ for all $k > N$. Since $s - \epsilon < \sup\{a_k : k \geq n\}$, then for every $n > N$, there exists $k_n > n$ such that $s - \epsilon < a_{k_n}$.
   "$\Longleftarrow$" Suppose that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n < s + \epsilon$ for all $n > N$ and for every $n > N$ there exists $k_n \geq n$ such that $s - \epsilon < a_{k_n}$. Then $\sup\{a_k : k \geq N\} < s + \epsilon$. Therefore $\sup\{a_k : k \geq n\} \leq \sup\{a_k : k \geq N\} < s + \epsilon$ for all $n > N$. Furthermore $s - \epsilon < \sup\{a_k : k \geq n\}$ since $s - \epsilon < a_{k_n}$. This means $|\sup\{a_k : k \geq n\} - s| < \epsilon$ for all $n > N$.
2. "$\Longleftarrow$" Suppose $\{a_n\}$ is unbounded above. Then for any given $n, m \in \mathbb{N}$, $a_n < \sup\{a_k : k \geq m\}$. This implies that for any $A > 0$,

   $$A < \sup\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n+1\}$$

   Therefore, $\limsup a_n = \infty$
   "$\Longrightarrow$" Suppose $\limsup a_n = \infty$. Then for any $A > 0$, there exists $N \in \mathbb{N}$ such that $\sup\{a_k : k \geq n\} > A$ for all $n > N$. Therefore there exists $k_n \geq n$ such that $a_{k_n} > A$ and the sequence $\{a_n\}$ is unbounded above.
3. Left as an exercise
4. "$\Longrightarrow$" Suppose $\lim_{n\to\infty} a_n = s$ with $s \in \mathbb{R}$ (the case $s = \pm\infty$ is left as an exercise). Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - s| < \epsilon$ for all $n > N$. Therefore

   $$s - \epsilon \leq \inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\} \leq s + \epsilon \quad \forall n > N\,.$$

   This means $\limsup a_n = \liminf a_n = s$.
   "$\Longleftarrow$" Left as an exrecise.

□

To a sequence of real numbers $\{a_n\}$ we associate the sequence of partial sums $\{s_n\}$ defined by $s_n = \sum_{j=1}^{n} a_j$. The series $\sum_{j=1}^{\infty} a_j$ converges to s if the sequence $\{s_n\}$ converges to s and we write $s = \sum_{j=1}^{\infty} a_j$.

Proposition

Let $\{a_n\}$ be a sequence of real numbers.

1. *$\sum_{j=1}^{\infty} a_j$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\left| \sum_{j=n}^{n+m} a_j \right| < \epsilon \quad \text{for } n \geq N \text{ and } m \in \mathbb{N}\,. \tag{$*$}$$

2. *If $\sum_{j=1}^{\infty} |a_j|$ converges then so does $\sum_{j=1}^{\infty} a_j$.*
3. *If $a_j \geq 0$ for all $j \in \mathbb{N}$, then $\sum_{j=1}^{\infty} a_j$ converges if and only the sequence of partial sums is bounded*

Proof.

1. Suppose $\sum a_j$ converges. Then the sequence of partial sums $\{s_n\}$ is Cauchy. Therefore for $\epsilon > 0$, there exists $N \in \mathbb{N}$ for any $q \geq p > N$ we have $|s_q - s_p| < \epsilon$. Set $n = p - 1$ and $q = n + m$ with $n \geq N$ and $m \in \mathbb{N}$. We have

$$|s_q - s_p| = \left| \sum_{j=1}^{n+m} a_j - \sum_{j=1}^{n-1} a_j \right| = \left| \sum_{j=n}^{n+m} a_j \right| < \epsilon .$$

   Conversely, suppose that $\sum a_j$ satisfies condition $(*)$. Then the sequence $\{s_n\}$ is Cauchy and so converges.

2. Suppose $\sum |a_j|$ converges, then by part (1) for any $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for all $n > N$ and $m \in \mathbb{N}$ we have $\sum_{j=n}^{n+m} |a_j| < \epsilon$. This implies $\left| \sum_{j=n}^{n+m} a_j \right| \leq \sum_{j=n}^{n+m} |a_j| < \epsilon$. Hence $\sum a_j$ converges by part (1).

3. If $a_j \geq 0$ for all j, the sequence of partial sums $\{s_n\}$ is increasing. Therefore $\{s_n\}$ converges if and only if it is bounded.

□