Real Analysis MAA 6616
Lecture 30
Hahn and Jordan Decompositions
Radon-Nikodym Theorem

Theorem (1)

Let $(X, \mathcal{A}, \mu)$ be a signed measure space. Then there exists positive and negative sets $P, N \in \mathcal{A}$ such that $P \cap N = \emptyset$ and $P \cup N = X$. If μ takes negative values, then $\mu(N) < 0$ and if $-\mu$ takes negative values $\mu(P) > 0$. Moreover we have uniqueness of the such a pair in the sense that if P', N' is any other such pair, then the symmetric difference sets $P \triangle P'$ and $N \triangle N'$ are null sets.

Proof.

If there are no μ-negative sets, then there is nothing to prove. Suppose then that there are negative sets. Let $r = \inf\{\mu(A) : A \text{ negative set}\}$ and $\{A_j\}_{j=1}^{\infty}$ be a sequence of negative sets such that $\mu(A_j) \to r$. Define $N = \bigcup_n A_n$ and $P = X \backslash N$. Then N and P are disjoint, $X = P \cup N$. We need to verify that N is negative and P positive.
Let $B_1 = A_1,\ B_2 = A_2 \backslash B_1$, in general let $B_k = A_k \backslash (B_1 \cup \cdots \cup B_{k-1})$. Note that since A_n is a negative set and $B_n \subset A_n$, then $\mu(B_n) \leq 0$; the B_n's are disjoint; and $\bigcup_n B_n = \bigcup_n A_n = N$. Let $E \subset N$, then $E \cap B_n \subset A_n$ and $\mu(E \cap B_n) \leq 0$. We have $E = \bigcup_n (E \cap B_n)$ and $\mu(E) = \sum_n \mu(E \cap B_n) \leq 0$. This means that N is a negative set.
Moreover, we can write $N = A_n \cup (N \backslash A_n)$ to get $\mu(N) \leq \mu(A_n) + \mu(N \backslash A_n) \leq \mu(A_n)$. By letting $n \to \infty$ we get $\mu(N) \leq r$. Therefore $\mu(N) = r$ and $r > -\infty$.
Now we need to verify that $P = X \backslash N$ is a positive set. By contradiction, if it were not, then there would exist $C \subset P$ such that $\mu(C) < 0$. By Proposition 2 (Lecture 29) there exists a negative set $N_1 \subset N$. Let $\tilde{N} = N \cup N_1$. Then $\tilde{N}$ is a negative set and $\mu(\tilde{N}) = \mu(N) + \mu(N_1) < r$ which is impossible. Therefore P is a positive set.
Now suppose that μ is not a positive measure, we need to verify that $\mu(N) < 0$. If $\mu(N) = 0$. Let $E \in \mathcal{A}$, then $E = (E \cap N) \cup (E \cap P)$ and we have

$$\mu(E) = \mu(E \cap N) + \mu(E \cap P) \geq 0 + \mu(E \cap P) \geq 0.$$

This means μ is a positive measure (a contradiction). Similar argument can be used to verify that if $-\mu$ is not a positive measure, then $\mu(P) > 0$.

To verify uniqueness of the pair N, P. Suppose N', P' is another such pair. Note that $N \backslash N' = P' \backslash P$ so that if $S \subset N \backslash N' = P' \backslash P$ then $\mu(S) \leq 0$ as a subset of a negative set and $\mu(S) \geq 0$ as a subset of a positive set. Therefore $\mu(S) = 0$. Any set $A \subset N \triangle N'$ can be written as $A = S \cup T$ with $S \subset N \backslash N'$ and $T \subset N' \backslash N$. Therefore $\mu(A) = \mu(S) + \mu(T) = 0$. □

Example

Let $f \in \mathcal{L}(\mathbb{R}^n, dm)$ (m is the Lebesgue measure) and let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$. Defined a signed measure μ on $\mathcal{M}$ by $\mu(E) = \int_E f dm$. Let $P = \{x \in \mathbb{R}^n : f(x) \geq 0\}$ and $N = \{x \in \mathbb{R}^n : f(x) < 0\}$. The pair P, N satisfies Proposition 1.

Two measures μ and ν defined on the same sigma algebra $\mathcal{A}$ are said to be mutually singular and denoted $\mu \perp \nu$ if there exist disjoint sets $A, B \in \mathcal{A}$ such that $X = A \cup B$, $\mu(A) = 0$ and $\nu(B) = 0$

Example

This example will establish that the Lebesgue measure and the Lebesgue-Stieltjes measure generated by the Cantor-Lebesgue function are mutually singular.

First define the Lebesgue-Stieltjes measure. Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function that is continuous from the right (i.e. $\lim_{x \to c^+} \alpha(x) = \alpha(c)$ for all $c \in \mathbb{R}$). For $a < b$ define $\nu((a,\ b]) = \alpha(b) - \alpha(a)$. The ν extends as a measure $\nu : \mathcal{M} \longrightarrow [0,\ \infty]$, where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$. The measure ν is called the Lebesgue-Stieltjes measure generated by the function α.

Now consider the function α given by $\alpha(x) = 0$ if $x \leq 0$, $\alpha(x) = 1$ if $x \geq 1$, and $\alpha(x) = \phi(x)$, if $0 \leq x \leq 1$, where $\phi : [0,\ 1] \longrightarrow [0,\ 1]$ is the Cantor-Lebesgue function. Let ν be the Lebesgue-Stieltjes measure generated by this function. Note that since α is constant on $(-\infty,\ 0]$ and on $[1,\ \infty)$, then $\nu(E) = 0$ if E is contained in $\mathbb{R} \backslash [0,\ 1]$.

If m is the Lebesgue measure on $\mathbb{R}$, then $m \perp \nu$. Indeed let $A = C$, where $C \subset [0,\ 1]$ is the Cantor set, and let $B = \mathbb{R} \backslash C$. We already now that $m(C) = 0$, we are left to verify that $\nu(\mathbb{R} \backslash C) = 0$. The set $\mathbb{R} \backslash C$ is a union of disjoint intervals:

$$\mathbb{R} \backslash C = (-\infty,\ 0) \cup \bigcup_{j=1}^{\infty} I_j \cup (1,\ \infty)$$

where I_j are the open middle third intervals removed from $[0,\ 1]$ in the construction of the Cantor set. Since ϕ is constant on each interval I_j, then $\nu(I_j) = 0$. We already noted that $\nu(-\infty,\ 0) = 0$ and $\nu(1,\ \infty) = 0$. Therefore $\nu(\mathbb{R} \backslash C) = 0$ and the two measures are mutually singular.

Theorem (2)

Let $\mu : \mathcal{A} \longrightarrow (-\infty, \infty]$ be a signed measure on a space X. Then there exist positive measures $\mu^+, \mu^- : \mathcal{A} \longrightarrow [0, \infty]$, with $\mu^+ \perp \mu^-$ and such that $\mu = \mu^+ - \mu^-$. Furthermore, this decomposition is unique.

Proof.

Let $P, N \in \mathcal{A}$ be such that P is a positive set, N is a negative set for μ, $P \cap N = \emptyset$ and $X = P \cup N$ (Hahn Decomposition Theorem). Define $\mu^\pm$ as follows. For $A \in \mathcal{A}$ set $\mu^+(A) = \mu(A \cap P)$ and $\mu^-(A) = -\mu(A \cap N)$. Then both μ^+ and μ^- are positive measures and $\mu = \mu^+ - \mu^-$. Moreover, we have $\mu^+(N) = \mu(P \cap N) = 0$ and $\mu^-(P) = \mu(P \cap N) = 0$ so that $\mu^+ \perp \mu^-$.
Now suppose that $\mu = \nu^+ - \nu^-$ is another such decomposition with $\nu^+ \perp \nu^-$. Let $S^+, S^- \in \mathcal{A}$ be the associated pair in the Hahn decomposition: $S^+ \cap S^- = \emptyset$, $X = S^+ \cup S^-$, $\nu^\pm(S^\mp) = 0$. It follows from the uniqueness of the Hahn decomposition that $P \triangle S^+$ and $N \triangle S^-$ are μ-null sets.

Let $A \in \mathcal{A}$. We have

$$\nu^+(A) = \nu^+(A \cap S^+) = \nu^+(A \cap S^+) - \nu^-(A \cap S^+) = \mu(A \cap S^+) = \mu(A \cap P) = \mu^+(A).$$

This shows that $\mu^+ = \nu^+$. A similar argument can be used to prove $\mu^- = \nu^-$. □

The measure $|\mu| : \mathcal{A} \longrightarrow [0, \infty]$ given by $|\mu| = \mu^+ + \mu^-$ is called the variation measure of μ and $|\mu|(A) = \mu^+(A) + \mu^-(A)$ the total variation of A.

Let $\mu,\ \nu$ be two measures defined in a σ-algebra over a set X. The measure ν is said to be absolutely continuous with respect to μ, if every μ-null set is also a ν-null set. That is

$$\forall A \in \mathcal{A}\ \ \mu(A) = 0 \implies \nu(A) = 0.$$

In this case we write $\nu \ll \mu$.

Proposition (1)

Let μ and ν be measures defined in $\mathcal{A}$ over X and such that ν is finite ($\nu(X) < \infty$). Then $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{A}$ with $\mu(A) < \delta$, we have $\nu(A) < \epsilon$.

Proof.

"$\Longleftarrow$" Let $A \in \mathcal{A}$ such that $\mu(A) = 0$, we need to show that $\nu(A) = 0$. Let $\epsilon > 0$ and $\delta > 0$ such that satisfies the condition of the proposition. Since $\mu(A) = 0 < \delta$, then $\nu(A) < \epsilon$. Since $\epsilon > 0$ is arbitrary, then $\nu(A) = 0$.

"$\Longrightarrow$" Suppose that $\nu \ll \mu$. By contradiction, suppose that there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$, there exists a set $A_n \in \mathcal{A}$ such that $\mu(A_n) < 2^{-n}$ and $\nu(A_n) > \epsilon_0$. Let $A = \limsup_n A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k$. Then

$$\mu(A) = \lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(A_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{2^k} = 0.$$

This means $\mu(A) = 0$ and $\nu(A) \geq \epsilon_0 > 0$ which is a contradiction. □

Proposition (2)

Let μ and ν be finite positive measures defined in $\mathcal{A}$ over X. Then either $\mu \perp \nu$ or else there exists $\epsilon > 0$ and a set $P \in \mathcal{A}$ such that $\mu(P) > 0$ and P is a positive set for the measure $\nu - \epsilon\mu$.

Proof.

Let $n \in \mathbb{N}$ and consider the signed measure $\nu - \frac{1}{n}\mu$. It follows from the Hahn decomposition that there exists a pair sets P_n (positive) and N_n (negative) such that $P_n \cap N_n = \emptyset$, $P_n \cup N_n = X$ for this measure.

Let $N = \bigcap_n N_n$ and $P = \bigcup_n P_n$. We have $X \backslash N = \bigcap_n (X \backslash N_n) = \bigcup_n P_n = P$. For each n we have $N \subset N_n$ and

$$0 \leq \nu(N) \leq \nu(N_n) \leq \frac{1}{n}\mu(N_n) \leq \frac{1}{n}\mu(X).$$

Since $\mu(X) < \infty$, then $\nu(N) = 0$. Now we have two possibilities. Either $\mu(P) = 0$ and then $\nu \perp \mu$ or else $\mu(P) > 0$. In this case there exists n_0 such that $\mu(P_{n_0}) > 0$. Let $\epsilon = 1/n_0$. Then from the definition of P_{n_0} we have

$\nu(P_{n_0}) - \epsilon\mu(P_{n_0}) > 0$ □

Remark (1)

We know from earlier examples that if f is a nonnegative μ-integrable function over X, then the set function $\nu : \mathcal{A} \longrightarrow \mathbb{R}^+$ given by $\nu(A) = \int_A f d\mu$ is a measure. The Radon-Nikodym gives a sufficient condition under which ν has the above form. The function f is called the derivative (or density) of ν with respect to μ and is denoted $f = \frac{d\nu}{d\mu}$ or $d\nu = f d\mu$

Theorem (3)

Let $\mu : \mathcal{A} \longrightarrow [0, \infty]$ be a σ-finite positive measure on X and let $\nu : \mathcal{A} \longrightarrow [0, \infty)$ be a finite measure on X such that ν is absolutely continuous with respect to μ ($\nu \ll \mu$). Then there exists a μ-integrable nonnegative function $f \in \mathcal{L}(X, \mu)$ such that

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A \in \mathcal{A}.$$

Moreover if $g \in \mathcal{L}(X, \mu)$ is any other such function, then $g = f$ a.e. in X.

Proof.

Uniqueness. If $f, g \in \mathcal{L}(X, \mu)$ are two such functions, then $h = f - g \in \mathcal{L}(X, \mu)$ and for every $A \in \mathcal{A}$ we have

$$\int_A h d\mu = \int_A (f - g) d\mu = \int_A f d\mu - \int_A g d\mu = \nu(A) - \nu(A) = 0$$

Hence $h = 0$ a.e. in X.

To prove the existence of the function f we consider two cases.

- Case 1: μ is a finite measure ($\mu(X) < \infty$). Consider the family of functions

$$\mathcal{F} = \left\{ h \in \mathcal{L}(X, \mu) : \ h \geq 0 \text{ and } \int_A h d\mu \leq \nu(A) \text{ for all } A \in \mathcal{A} \right\}$$

  Note that $\mathcal{F} \neq \emptyset$ since $0 \in \mathcal{F}$ and if $\alpha, \beta \in \mathcal{F}$ then $\max(\alpha, \beta) \in \mathcal{F}$. Indeed, consider the sets

$$C = \{x : \ \alpha(x) \geq \beta(x)\} \text{ and } D = \{x : \ \alpha(x) < \beta(x)\} = X \backslash C$$

  so that $C \cap D = \emptyset$ and $C \cup D = X$. Let $A \in \mathcal{A}$. Then

$$\int_A \max(\alpha, \beta) d\mu = \int_{A \cap C} \alpha d\mu + \int_{A \cap D} \beta d\mu \leq \nu(A \cap C) + \nu(A \cap D) = \nu(A) \qquad .$$

  This implies $\max(\alpha, \beta) \in \mathcal{F}$.

□

Proof.

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Let $M = \sup\left\{\int_X h d\mu \,:\, h \in \mathcal{F}\right\}$. Note that since $\int_X h d\mu \leq \nu(X) < \infty$, then $M < \infty$. Let $\{h_n\}_n \subset \mathcal{F}$ such that $\lim_{n\to\infty} \int_X h_n d\mu = M$. For each $n \in \mathbb{N}$, let $f_n = \max(h_1, \cdots, h_n)$. Then $f_n \in \mathcal{F}$ and $\{f_n\}_n$ is an increasing sequence. Let $f = \lim_n \to \infty f_n$. It follows from the Monotone Convergence Theorem that for every $A \in \mathcal{A}$ we have

$$\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu \leq \nu(A).$$

Hence $f \in \mathcal{F}$. Moreover

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu \geq \lim_{n\to\infty} \int_X h_n d\mu = M.$$

Therefore $\int_X f d\mu = M$.

We are left to verify that $\int_A f d\mu = \nu(A)$. For this we define a positive measure $\lambda : \mathcal{A} \longrightarrow [0, \infty)$ by

$$\lambda(A) = \nu(A) - \int_A f d\mu.$$

We claim that $\lambda \perp \mu$. By contradiction, if λ were not mutually singular to μ, then (Proposition 2) there exists $\epsilon > 0$ and $P \in \mathcal{A}$ such that $\mu(P) > 0$ and P is a positive set for the measure $\lambda - \epsilon\mu$. Let $A \in \mathcal{A}$. We have

$$\nu(A) - \int_A f d\mu = \lambda(A) \geq \lambda(A \cap P) \geq \epsilon\mu(A \cap P) = \int_A \epsilon\chi_P d\mu.$$

Then $\nu(A) \geq \int_A (f + \epsilon\chi_P) d\mu$. This means $h = f + \epsilon\chi_P \in \mathcal{F}$. But,

$$\int_X h d\mu = \int_X f d\mu + \epsilon \int_X \chi_P d\mu = M + \epsilon\mu(P) > M.$$

This is a contradiction and so $\lambda \perp \mu$.

□

Proof.

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Since $\lambda \perp \mu$, then there exists $Z \in \mathcal{A}$ such that $\mu(Z) = 0$ and $\lambda(X \backslash Z) = 0$. It follows from $\nu \ll \mu$ that $\nu(Z) = 0$. Therefore $\lambda(Z) = \nu(Z) - \int_Z f d\mu = 0$. We have then $\lambda(X) = \lambda(Z) + \lambda(X \backslash Z) = 0$. So that $\lambda = 0$. Therefore $\nu(A) = \int_A f d\mu$ for $A \in \mathcal{A}$. This completes the proof when μ is finite.

- Case 2: μ is σ-finite. In this case there exists a sequence $\{X_j\}_j \subset \mathcal{A}$ such that $X_j \nearrow X$ and $\mu(X_j) < \infty$ for all j. For $j \in \mathbb{N}$, Let μ_j and ν_j be the restrictions of μ and ν to X_j: $\mu_j(A) = \mu(A \cap X_j)$ and $\nu_j(A) = \nu(A \cap X_j)$ for $A \in \mathcal{A}$. Then μ_j is a finite measure and $\nu_j \ll \mu_j$. Indeed, if $\mu_j(E) = 0$, then $\mu(E \cap X_j) = 0$ and so $0 = \nu(E \cap X_j) = \nu_j(E)$.
  It follows from the first case that there exists $f_j \in \mathcal{L}(X, \mu_j)$ such that $d\mu_j = f_j d\mu_j$. It follows from the uniqueness of the density that $f_i = f_j$ a.e. in X if $i \leq j$. Define $f \in \mathcal{L}(X, \mu)$ by $f(x) = f_j(x)$ if $x \in X_j$.
  Let $A \in \mathcal{A}$, we have
  $$\nu(A) = \lim_{j \to \infty} \nu(A \cap X_j) = \lim_{j \to \infty} \nu_j(A) = \lim_{j \to \infty} \int_A f_j d\mu_j = \lim_{j \to \infty} \int_{A \cap X_j} f d\mu = \int_A f d\mu.$$
  This completes the proof.

□

Theorem (4)

Let μ and ν be respectively σ-finite and finite positive measures defined over a σ-algebra $\mathcal{A}$ of a space X. Then there exist unique positive measures λ and ρ such that $\nu = \lambda + \rho$, with $\rho \ll \mu$ and $\lambda \perp \mu$.

Remark (2)

This theorem is known as the Lebesgue Decomposition Theorem. Its proof is analogous to that of the Radon-Nikodym Theorem. What is missing here is $\nu \ll \mu$. Once $\mathcal{F}$, M, and f are defined exactly as in the proof of the Radon-Nikodym Theorem, we can define ρ by

$\rho(A) = \int_A f d\mu$ and $\lambda = \nu - \rho$.