Real Analysis MAA 6616
Lecture 31
The Dual Space of L^p

Recall (from Lecture 26) that if $(X, \|\cdot\|)$ be a normed space. A linear functional on X is a map $T : X \longrightarrow \mathbb{R}$ such that $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$.

The linear functional T is said to be bounded if there exists $M > 0$ such that $T(f) \leq M \|f\|$ for all $f \in X$. Denote by X^* the space of all bounded linear functionals on X. the space X^* is called the dual of X. For linear functional $T \in X^*$, define $\|T\|_*$ by

$$\|T\|_* = \inf\{M : |Tf| \leq M \|f\| \text{ for all } f \in X.\} = \sup\{|Tf| : f \in X \text{ with } \|f\| \leq 1.\}$$

If $(X, \|\cdot\|)$ is Banach space, then so is $(X^*, \|\cdot\|_*)$ (Theorem 1 of Lecture 26).

We are interested in characterizing the dual space of $L^p(X, \mu)$ for $1 \leq p < \infty$ and μ a σ-finite measure. This is given by the Riesz Representation Theorem that says that $L^p(X, \mu)^* \cong L^q(X, \mu)$ where $\frac{1}{q} + \frac{1}{q} = 1$. First recall the following

Proposition (1)

Let $1 \leq p < \infty$, q the conjugate of p, E a μ-measurable set and $g \in L^q(E, \mu)$. Then the $T_g : L^p(E, \mu) \longrightarrow \mathbb{R}$ defined by $T_g(f) = \int_E fg d\mu$ is a bounded linear functional and $\|T_g\|_* \leq \|g\|_q$.

The proof is a direct consequence of the linearity of the integral and of the Hölder inequality.

Theorem (1)

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 \leq p < \infty$, and q the conjugate of p. Then the map

$$\Phi : L^q(X, \mu) \longrightarrow L^p(X, \mu)^*$$

given by $\Phi(g) = T_g$ is an isometric isomorphism, where $T_g : L^p(X, \mu) \longrightarrow \mathbb{R}$, $T_g(f) = \int_E fg d\mu$. Isometric isomorphism means that Φ is linear, bijective, and $\|\Phi(g)\|_* = \|g\|_q$ for all $g \in L^q(X, d\mu)$.

Proof.

We already know from Proposition 1 that the map Φ is well defined and that $\|\Phi(g)\|_* \leq \|g\|_q$ for all $g \in L^q(X, d\mu)$. To continue, we need consider two cases on whether or not the measure μ is finite.

- Case $\mu(X) < \infty$. Let $T \in L^p(X, \mu)^*$. Define a set function

  $$\nu : \mathcal{A} \longrightarrow \mathbb{R} \text{ by } \nu(A) = T(\chi_A) \text{ for } A \in \mathcal{A}.$$

  Note that this set function is well defined since $\mu(X) < \infty$ implies $\chi_A \in L^p(X, d\mu)$. Now we prove that ν is a countably additive signed measure. First we verify finite additivity. If $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 = \emptyset$, then

  $$\nu(A_1 \cup A_2) = T(\chi_{A_1 \cup A_2}) = T(\chi_{A_1} + \chi_{A_2}) = T(\chi_{A_1}) + T(\chi_{A_2}) = \nu(A_1) + \nu(A_2).$$

  To verify the countable additivity, let $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$ be a disjoint collection. Let $A = \bigcup_j A_j$ and write $A = B_n \cup C_n$, where $B_n = \bigcup_{j=1}^{n} A_j$ and $C_n = \bigcup_{j=n+1}^{\infty} A_j$. Then $B_n \cap C_n = \emptyset$ and

  $$\nu(A) = \nu(B_n \cup C_n) = \nu(B_n) + \nu(C_n) = \sum_{j=1}^{n} \nu(A_j) + \nu(C_n).$$

  Since $\mu(A) \leq \mu(X) < \infty$ and μ is countably additive, then $\mu(C_n) \to 0$ as $n \to \infty$. It follows from $|\nu(C_n)| \leq \|T\|_* \left\|\chi_{C_n}\right\|_p = \|T\|_* \mu(C_n)^{1/p}$ that $\lim_{n\to\infty} \nu(C_n) = 0$. Hence $\nu(A) = \sum_{j=1}^{\infty} \nu(A_j)$ This shows that ν is a signed measure defined on $\mathcal{A}$.

  Note also that if follows from $\nu(A) \leq \|T\|_* \mu(A)^{1/p}$ that $\nu(A) = 0$ whenever $\mu(A) = 0$. This means that $\nu \ll \mu$ (i.e. ν absolutely continuous with respect to μ).

Proof.

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Now we can apply the Radon-Nikodym Theorem: There exists a unique $g \in L^1(X, d\mu)$ such that

$$T(\chi_A) = \nu(A) = \int_A g d\mu = \int_X \chi_A g d\mu \quad \text{for all} \quad A \in \mathcal{A}.$$

Let ϕ be any μ-measurable simple function: $\phi = \sum_{j=1}^n c_j \chi_{A_j}$ for some $c_1, \cdots, c_n \in \mathbb{R}$ and disjoint sets $A_1, \cdots, A_n \in \mathcal{A}$. Then

$$T(\phi) = \sum_{j=1}^n c_j T(\chi_{A_j}) = \sum_{j=1}^n c_j \int_X \chi_{A_j} g d\mu = \int_X \phi g d\mu.$$

Now we need to extend this formula and show that $T(f) = \int_X f g d\mu$ for all $f \in L^p(X, d\mu)$. Before this we establish that $g \in L^q(X, d\mu)$ and $\|g\|_q = \|T\|_*$. Note since $\|g\|_q \geq \|T\|_*$ (Hölder's inequality), it is enough to establish. $\|g\|_q \leq \|T\|_*$. If $g = 0$, there is nothing to prove. Suppose $g \neq 0$.

Consider the case $p > 1$. Since $|g|^q \in L^1(X, d\mu)$ there exists a sequence of simple functions $\{\phi_k\}_k \subset L^1(X, d\mu)$ such that $0 \leq \phi_k$ and $\phi_k \nearrow |g|^q$.

For each k, consider the simple function $\psi_k = \phi_k^{\frac{1}{p}} \operatorname{sgn}(g)$. Then $\psi_k \in L^p(X, d\mu)$ and $\|\psi_k\|_p = \|\phi_k\|_1^{\frac{1}{p}}$. We have

$$T(\psi_k) = \int_X \psi_k g d\mu = \int_X \phi_k^{\frac{1}{p}} |g| d\mu \leq \|T\|_* \|\psi_k\|_p = \|T\|_* \|\phi_k\|_1^{\frac{1}{p}}.$$

Next, since $\phi_k^{\frac{1}{q}} \nearrow |g|$ and $\psi_k g = \phi_k^{\frac{1}{p}} |g| \geq \phi_k^{\frac{1}{p}+\frac{1}{q}} = \phi_k$, then

$$\|\phi_k\|_1 = \int_X \phi_k d\mu \leq \int_X \psi_k g d\mu = T(\psi_k) \leq \|T\|_* \|\psi_k\|_p = \|T\|_* \|\phi_k\|_1^{\frac{1}{p}}.$$

It follows that $\|\phi_k\|_1^{1-\frac{1}{p}} \leq \|T\|_*$ and as $k \to \infty$ we get $\|g\|_q \leq \|T\|_*$.

□

Proof.

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To complete the proof in the case $p > 1$, it remains to verify the uniqueness of g. Suppose there exists $g' \in L^q(X, d\mu)$ such that $T(f) = \int_X fg' d\mu$ for all $f \in L^p(X, d\mu)$. Let $A \in \mathcal{A}$, then

$$0 = T(\chi_A) - T(\chi_A) = \int_X \chi_A g d\mu - \int_X \chi_A g' d\mu = \int_X \chi_A (g - g')d\mu = \int_A (g - g')d\mu \,.$$

Since $A \in \mathcal{A}$ is arbitrary, then $g = g'$ a.e. in X.
Now we consider the case $p = 1$ so that $g \in L^\infty(X, d\mu)$. We need to verify that $\|T\|_*$ is an essential upper bound for $|g|$. By contradiction $\|T\|_* < \|g\|_\infty$. Then there exists $\epsilon > 0$ such that the set $S_\epsilon = \{x \in X : |g(x)| > \|T\|_* + \epsilon\}$ has positive measure. Without loss of generality, we can assume that $\mu(S^1_\epsilon) > 0$, where $S^1_\epsilon = \{x \in X : g(x) > \|T\|_* + \epsilon\}$. Then

$$\|T\|_* \mu(S^1_\epsilon) \geq \left|T(\chi_{S^1_\epsilon})\right| = \int_{S^+_\epsilon} gd\mu > (\|T\|_* + \epsilon)\mu(S^1_\epsilon).$$

This is contradiction. This completes the proof of the theorem when μ is finite.

- Case $\mu(X) = \infty$. Since μ is σ-finite, there exists an ascending sequence $\{X_j\}_j \subset \mathcal{A}$ such that $X_j \nearrow X$ and $\mu(X_j) < \infty$ for all j. Note that $L^p(X_j, d\mu)$ can be identified as a subset of $L^p(X, d\mu)$: an element $f \in L^p(X_j, d\mu)$ can considered as $\tilde{f} \in L^p(X, d\mu)$, where $\tilde{f}$ is the extension of f by setting $\tilde{f} = 0$ on $X \backslash X_j$ and we have $\|f\|_{p,X_j} = \|f\|_{f,X}$. Now let $T \in L^p(X, d\mu)^*$. For each $j \in \mathbb{N}$, let T_j be the restriction of T to $L^p(X_j, d\mu)^*$: $T_j(f) = T(\tilde{f})$. We have $\|T_j\|_* \leq \|T\|_*$.
  Since $\mu(X_j) < \infty$ it follows from the first case, that for each j there exists a unique $g_j \in L^q(X_j, d\mu)$ such that $T_j(f) = \int_{X_j} fg_j d\mu$, for all $f \in L^p(X_j, d\mu)$, and $\|g\|_{q,X_j} = \|T_j\|_* \leq \|T\|_*$. Since $X_j \subset X_{j+1}$, then for every $f \in L^p(X_j, d\mu)$, we have $T_j(f) = T(f) = T_{j+1}(f)$. This implies that $g_j = g_{j+1}$ a.e. in X_j. Without loss of generality we can assume $g_j = g_{j+1}$ on X_j and define a function g on X by $g(x) = g_j(x)$ if $x \in X_j$. Then g is μ-measurable and $\|g\|_q \leq \|T\|_*$. Let $f \in L^p(X, d\mu)$. Then $fg \in L^1(X, d\mu)$ (follows from Hölder's Inequality).
  We have $T(f\chi_{X_j}) = T_j(f) = \int_{X_j} fg_j d\mu = \int_{X_j} fgd\mu$. Since $f\chi_{X_j} \to f$ in $L^p(X, d\mu)$, then $\int_{X_j} fgd\mu = T(f\chi_{X_j}) \to \int_X fgd\mu = T(f)$. This completes the proof.

In general $L^\infty(X, d\mu)^* \neq L^1(X, d\mu)$: The dual space of $L^\infty(X, d\mu)$ cannot be identified with $L^1(X, d\mu)$. The Dirac δ functional provides an example of a functional on L^∞ that cannot be represented as $\delta(f) = \displaystyle\int fg$ with $g \in L^1$. The example below relies on the Hahn-Banach Theorem: *Let* $(X, \|\cdot\|)$ *be a linear normed space and* $Y \subset X$ *be a linear subspace. Let* $S \in Y^*$ *(i.e.* $S : Y \longrightarrow \mathbb{R}$ *a linear and bounded operator with norm* $\|S\|_* < \infty$*). Then there a bounded linear operator* $T \in X^*$ *such that* $T = S$ *on* Y *and* $\|T\|_* = \|S\|_*$

Example

The space $C^0([-1,\ 1])$ is a linear subspace of $L^\infty[-1,\ 1]$. The linear functional

$$\delta : C^0([-1,\ 1]) \longrightarrow \mathbb{R}$$

given by $\delta(f) = f(0)$ extends (Hahn-Banach Theorem) as an element of $L^\infty[-1,\ 1]^*$. We claim that there is no function $g \in L^1[-1,\ 1]$ such that $\delta(f) = \displaystyle\int_{-1}^{1} fg dx$. If there were such g, let $\epsilon > 0$ and let A be any measurable subset of $[-1,\ 1] \backslash [-\epsilon,\ \epsilon]$. For $f = \chi_A \in L^\infty[-1,\ 1]$ we have $0 = \delta(f) = \int_A g dx$. Since A is arbitrary, then $g = 0$ a.e. and this implies $\delta = 0$ a contradiction.