Real Analysis MAA 6616
Lecture 4
Continuous Functions

Let $E \subset \mathbb{R}$. A function $f : E \longrightarrow \mathbb{R}$ is said to be continuous at a point $a \in E$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in E$ such that $|x - a| < \delta$. f is continuous on E if it is continuous at all points of E. The function f is said to be Lipschitz on E if there exists a constant $L \geq 0$ (called Lipschitz constant) such that

$$|f(y) - f(x)| \leq L\,|y - x| \quad \text{for all } x, y \in E\,.$$

- Note that if $f : E \longrightarrow \mathbb{R}$ is Lipschitz, then it is continuous on E. Indeed, if L is the Lipschitz constant of f, then for $a \in E$ and $\epsilon > 0$, let $\delta = \epsilon/L$. For $x \in E$ such that $|x - a| < \delta$, we have $|f(x) - f(a)| \leq L\,|x - a| < L\delta = \epsilon$.
- A continuous function is not necessarily Lipschitz. For example $f(x) = \sqrt{x}$ is continuous on $[0,\ 1]$ but it is not Lipschitz. If it were Lipschitz with constant L, then we would have $\left|\sqrt{y} - \sqrt{x}\right| \leq L\,|y - x|$ for all $x, y \in [0,\ 1]$. This implies $1 \leq L(\sqrt{y} + \sqrt{x})$ for all $x, y \in [0,\ 1]$ (with $x \neq y$). A contradiction.

Proposition

A function $f : E \longrightarrow \mathbb{R}$ is continuous at $a \in E$ if and only if for every sequence $\{x_n\} \subset E$ such that $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof.

"$\Longrightarrow$" Suppose that f is continuous at $a \in \mathbb{R}$ and $\{x_n\} \subset E$ converges to a. Let $\epsilon > 0$. It follows from the continuity of f at a that there exists $\delta > 0$ such that $|f(y) - f(a)| < \epsilon$ for all $y \in E$ such that $|y - a| < \delta$. It follows from the convergence of $\{x_n\}$ that for given $\delta > 0$, there is $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n > N$. Therefore $|f(x_n) - f(a)| < \epsilon$ for all $n > N$.

"$\Longleftarrow$" Left as an exercise. □

Proposition

A function $f : E \longrightarrow \mathbb{R}$ is continuous on E if and only if for every open set $U \subset \mathbb{R}$ there is an open set $V \subset \mathbb{R}$ such that $f^{-1}(U) = E \cap V$.

Proof.

"$\Longrightarrow$" Suppose f is continuous on E. Let $U \subset \mathbb{R}$ be open and let $a \in f^{-1}(U)$. Since $f(a) \in U$ and U is open, then for $\epsilon > 0$ (small enough), we have $(f(a) - \epsilon,\ f(a) + \epsilon) \subset U$ and the continuity of f at a implies the existence of $\delta_a > 0$ such that for any $x \in E$ with $|x - a| < \delta_a$, we have $|f(x) - f(a)| < \epsilon$. Hence

$$f\left((a - \delta_a,\ a + \delta_a) \cap E\right) \ \subset\ (f(a) - \epsilon,\ f(a) + \epsilon) \ \subset\ U\,.$$

Define $V = \displaystyle\bigcup_{a \in f^{-1}(U)} (a - \delta_a,\ a + \delta_a)$. V is an open set (union of open intervals) and by construction we have $f^{-1}(U) \subset E \cap V$.

"$\Longleftarrow$" Suppose that for every open set $U \subset \mathbb{R}$ there exists an open set $V \subset \mathbb{R}$ such that $f^{-1}(U) = E \cap V$. We need to show that f is continuous on E. Let $a \in E$, let $\epsilon > 0$, and let $U = (f(a) - \epsilon,\ f(a) + \epsilon)$. Since U is open, then there exists an open set V such that

$$f^{-1}\left((f(a) - \epsilon,\ f(a) + \epsilon\right) \ =\ E \cap V\,. \qquad (*)$$

Since $a \in V$ and V is open, then there exists $\delta > 0$ such that $(a - \delta,\ a + \delta) \subset V$. It follows from $(*)$ that for $x \in E$ such that $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. Therefore f is continuous at a. □

Extreme Value Theorem

A function f is said to have a maximum value (respectively minimum value) on a set S if there exist $s^* \in S$ such that $f(x) \leq f(s^*)$ (respectively $f(x) \geq f(s^*)$) for all $x \in S$.

A function f is said to be bounded on a set S if the image $f(S)$ is a bounded set. Equivalently, if there exists $M \geq$ such that $|f(s| \leq M$ for all $s \in S$.

Theorem

Let $E \subset \mathbb{R}$ be closed and bounded and $f : E \longrightarrow \mathbb{R}$ be continuous. Then f has a maximum and a minimum values on E.

Proof.

We first prove that f is bounded. By contradiction, suppose that for every $n \in \mathbb{N}$ there is $x_n \in E$ such that $|f(x_n)| > n$. Since E is closed and bounded, then the sequence $\{x_n\}$ is bounded so has a convergent subsequence $\{x_{n_k}\}$. Let $x^* = \lim_{k\to\infty} x_{n_k}$. Then $x^* \in E$ (since E is closed and $\{x_{n_k}\} \subset E$.) The function f is continuous at x^*. Then for $\epsilon = 1$ there exists $\delta > 0$ such that for every $y \in E$ with $\left|y - x^*\right| < \delta$ we have $\left|f(y) - f(x^*)\right| < 1$ and so $|f(y)| \leq 1 + \left|f(x^*)\right|$. In particular there exists $N \in \mathbb{N}$ such that for $k > N$ we have $\left|x_{n_k} - x^*\right| < \delta$ and so $\left|f(x_{n_k}\right| \leq 1 + \left|f(x^*)\right|$. This is a contradiction since $\left|f(x_{n_k})\right| > n_k$ and n_k is a strictly increasing sequence. The function f is therefore bounded.
Let $M = \sup f(E)$ and $m = \inf f(E)$. We have $m, M \in \mathbb{R}$ since f is bounded. For $n \in \mathbb{N}$, $M - 1/n$ is not an upper bound of $f(E)$ and $m + 1/n$ is not a lower bound for $f(E)$. Therefore there exist $a_n \in E$ such that $M - 1/n < f(a_n) \leq M$ and there exist $b_n \in E$ such that $m \leq f(b_n) < m + 1/n$. Since E is bounded, then the sequences $\{a_n\}$ and $\{b_n\}$ have convergent subsequences. Let $a^* = \lim_{k\to\infty} a_{n_k}$ and $b_* = \lim_{k\to\infty} b_{n_k}$. The limits a^* and b_* are in E (E closed). The continuity of f at a^* and b_* imply that for any $\epsilon > 0$ and for $k \in \mathbb{N}$ large enough we have

$$M - \frac{1}{n_k} - \epsilon < f(a_{n_k}) - \epsilon < f(a^*) \leq M \text{ and } m \leq f(b_*) < f(b_{n_k}) + \epsilon < m + \frac{1}{n_{k_n}} + \epsilon$$

This implies that $f(a^*) = M$ is the maximum and $f(b_*) = m$ is the minimum value of f on E. □

Theorem

Let $[a,\ b]$ be a closed and bounded interval and let $f : [a,\ b] \longrightarrow \mathbb{R}$ be a continuous function. If $r \in \mathbb{R}$ is between $f(a)$ and $f(b)$, then there exists $x^* \in (a,\ b)$ such that $f(x^*) = r$.

Proof.

Suppose that $f(a) < f(b)$ so that $f(a) < r < f(b)$. Define sequences $\{a_n\}$ and $\{b_n\}$, contained in the interval $[a,\ b]$, inductively as follows: Set $a_1 = a$, $b_1 = b$. Let $m_1 = (a_1 + b_1)/2$ be the midpoint. Define $a_2 = a_1$ and $b_2 = m_1$ if r is between $f(a_1)$ and $f(m_1)$, otherwise define $a_2 = m_1$ and $b_2 = b_1$. Suppose that $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are defined. Let $m_n = (a_n + b_n)/2$. Set $a_{n+1} = a_n$ and $b_{n+1} = m_n$ if r is between $f(a_n)$ and $f(m_n)$ otherwise set $a_{n+1} = m_n$ and $b_{n+1} = b_n$. We have then a sequence of nested intervals

$$\cdots \subset [a_{n+1},\ b_{n+1}] \subset [a_n,\ b_n] \subset \cdots \subset [a_1,\ b_1]$$

such that $b_{n+1} - a_{n+1} = (b_n - a_n)/2$ and $f(a_n) \leq r \leq f(b_n)$. It follows from the Nested Set Theorem that the intersection of these intervals is nonempty. Let $x^* \in \bigcap_{j=1}^{\infty}[a_j,\ b_j]$. We have $x^* - a_j \leq b_j - a_j = \dfrac{b-a}{2^{j-1}}$. Therefore $a_j \longrightarrow x^*$. Similarly $b_j \longrightarrow x^*$. It follows from the continuity of f that $\lim_{j\to\infty} f(a_j) = f(x^*) = \lim_{j\to\infty} f(b_j)$. Since (by construction) $f(a_j) \leq r \leq f(b_j)$ for all j, then $r = f(x^*)$. □

Uniform Continuity

A function $f : E \longrightarrow \mathbb{R}$ is said to be uniformly continuous on E if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for every $x, y \in E$ such that $|y - x| < \delta$.

Theorem

Let $E \subset \mathbb{R}$ be closed and bounded and $f : E \longrightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous on E.

Proof.

Let $\epsilon > 0$, we need to find $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ whenever $x, y \in E$ satisfy $|y - x| < \delta$. Since f is continuous, then for every $a \in E$, there exists $\delta_a > 0$ such that $|f(x) - f(a)| < \epsilon/2$ for every $x \in E$ satisfying $|x - a| < \delta_a$. For every $a \in E$, define the interval $I_a = (a - \frac{\delta_a}{2},\ a + \frac{\delta_a}{2})$. The family $\{I_a\}_{a \in E}$ is then an open cover of the set E. Since E is closed and bounded, then it follows from the Heine Borel Theorem that E has a finite subcover. That is there exist $a_1, \cdots, a_n \in E$ such that $E \subset I_{a_1} \cup \cdots \cup I_{a_n}$

Let $\delta = \frac{1}{2} \min(\delta_{a_1}, \cdots, \delta_{a_n})$. Let $x, y \in E$ such that $|y - x| < \delta$. Since $x \in E$, then there exists $i \in \{1, \cdots, n\}$ such that $x \in I_{a_i}$ (i.e. $|x - a_i| < \frac{\delta_{a_i}}{2}$). We have also

$$|y - a_i| = |y - x + x - a_i| \le |y - x| + |x - a_i| < \delta + \frac{\delta_{a_i}}{2} \le \delta_{a_i}.$$

It follows then from the definition of δ_{a_i} that $|f(x) - f(a_i)| < \epsilon/2$ and $|f(y) - f(a_i)| < \epsilon/2$. Therefore

$$|f(y) - f(x)| \le |f(y) - f(a_i)| + |f(x) - f(a_i)| < \epsilon$$

□

A function $f : E \longrightarrow \mathbb{R}$ is said to be increasing (respectively, decreasing) on E if $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$) for all $x, y \in E$ with $x \leq y$. The function f is said to be monotone on E if it is either an increasing function on E or a decreasing function on E.

Proposition

Let $I \subset \mathbb{R}$ be an open interval, $x_0 \in I$, and $f : I \longrightarrow \mathbb{R}$ a monotone function.

1. *There exists $M \in \mathbb{R}$ such that for every decreasing sequence $\{x_n\}$ in I with $x_n > x_0$, and converging to x_0, $\lim_{n\to\infty} f(x_n) = M$, M is denoted by $f(x_0^+)$;*
2. *There exists $m \in \mathbb{R}$ such that for every increasing sequence $\{x_n\}$ in I, with $x_n < x_0$, and converging to x_0, $\lim_{n\to\infty} f(x_n) = m$, m is denoted by $f(x_0^-)$.*
3. *f is continuous at x_0 if and only if $f(x_0^-) = f(x_0) = f(x_0^+)$.*

- When f is as in the proposition and fails to be continuous at x_0, then $f(x_0)$ is the only value of $f(I)$ that is between $f(x_0^+)$ and $f(x_0^-)$ and f said to have a jump discontinuity at x_0.
- If $I \subset \mathbb{R}$ is an interval and $f : I \longrightarrow \mathbb{R}$ is monotone and continuous, then $f(I)$ is an interval. (This follows from the proposition and the Intermediate Value Theorem).

Proof.

We prove part (1) when f is increasing and leave the rest as an exercise. Let $\{x_n\}$ be a decreasing sequence in I with limit x_0 and $x_n > x_0$. Then the sequence $\{f(x_n)\}$ is decreasing and bounded $f(x_1) \geq f(x_n) \geq f(x_0)$ (f increasing). Therefore $\{f(x_n)\}$ converges (monotone convergence theorem for sequences) to a number $M \geq f(x_0)$.

Now we need to verify that M is independent on the sequence $\{x_n\}$. Let $\{x'_n\}$ be another decreasing sequence in I with limit x_0 and $x'_n > x_0$ and such that $x_k \neq x'_k$ for infinitely many natural numbers k. The previous argument shows that there exists $M' \geq f(x_0)$ such that $\{f(x'_n)\}$ converges to M'. We need to show that $M = M'$. For this we construct a new sequence $\{x''_n\}$ as follows: Define $x''_1 = x_1$. Let n_1 be the smallest integer such that $x'_{n_1} < x''_1$, define $x''_2 = x'_{n_1}$. By induction, suppose x''_j is defined for $j = 1, \cdots, k$. If $k = 2p + 1$ is odd, let n_k be the smallest integer such that $x'_{n_k} < x''_k$, and define $x''_{k+1} = x'_{n_k}$; If $k = 2p$ is even, let n_k be the smallest integer such that $x_{n_k} < x''_k$, and define $x''_{k+1} = x_{n_k}$. We have defined a decreasing sequence $\{x''_n\}$ with limit x_0 such that $\{x''_{2j+1}\}$ is a subsequence of $\{x_n\}$ and $\{x''_{2j}\}$ is a subsequence of $\{x'_n\}$. Since all these sequence converge and $\lim_{j\to\infty} f(x''_{2j+1}) = M$ and $\lim_{j\to\infty} f(x''_{2j}) = M'$, then $M = M'$.

□