Real Analysis MAA 6616
Lecture 5
Lebesgue Outer Measure

Let $I \subset \mathbb{R}$ be an interval. The length $\ell(I)$ is defined to be $\ell(I) = \infty$ if I is unbounded and $\ell(I) = b - a$ if I has finite left endpoint a and finite right endpoint b.
Let $E \subset \mathbb{R}$ be any set in $\mathbb{R}$ and let $\{I_j\}_{j=1}^{\infty}$ be a collection of bounded open intervals that cover the set E. That is $E \subset \bigcup_{j=1}^{\infty} I_j$. We associate to such a cover $\sum_{j=1}^{\infty} \ell(I_j)$ (which could be ∞). Define the outer measure of E as

$$m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \ell(I_j) : E \subset \bigcup_{j=1}^{\infty} I_j \right\}.$$

The following properties follow directly from the definition.

- $m^*(\emptyset) = 0$
- The outer measure is monotone. That is, if $E \subset F$, then $m^*(E) \leq m^*(F)$

Proposition

If $C \subset \mathbb{R}$ is countable, then $m^*(C) = 0$.

Proof.

Suppose $C = \{x_n\}$. Let $\epsilon > 0$. For each n, let $I_n = \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\right)$. Then $C \subset \bigcup_{n=1}^{\infty} I_n$ and $\ell(I_n) = \frac{\epsilon}{2^n}$.

Hence $\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$. Therefore $m^*(C) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $m^*(C) = 0$. □

Proposition

Let $I \subset \mathbb{R}$ be an interval. Then $m^*(I) = \ell(I)$. In particular $m^*(\mathbb{R}) = \infty$.

Proof.

• *Case $I = [a,\ b]$, closed and bounded.* Let $\epsilon > 0$. Then $I \subset (a-\epsilon,\ b+\epsilon)$ and it follows from the definition of outer measure that $m^*(I) \leq (b+\epsilon) - (a-\epsilon) = \ell(I) + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, then $m^*(I) \leq \ell(I)$.
To prove the proposition in this case, we need to prove that $m^*(I) \geq \ell(I)$. First suppose that I covered by finitely many open and bounded interval $I_1, \cdots, I_n$. We are going to show that $\sum_{j=1}^{n} \ell(I_j) \geq \ell(I)$. Since $a \in I$, then there exists $j_1 \in \{1, \cdots, n\}$ such that $a \in I_{j_1} = (a_1,\ b_1)$. If $b \leq b_1$, then $\ell(I) < \ell(I_{j_1})$ and we are done. If not, then $a < b_1 < b$, so that $b_1 \in I$ and there is $j_2 \in \{1, \cdots, n\}$ such that $b_1 \in I_{j_2} = (a_2,\ b_2)$. If $b \leq b_2$, then $\ell(I) < \ell(I_{j_1}) + \ell(I_{j_2})$. If not, we can repeat this process which must end after at most k steps with $k \leq n$ to get

$$\ell(I) < \sum_{p=1}^{k} \ell(I_{j_p}) < \sum_{j=1}^{n} \ell(I_j)$$

Now if $\{I_p\}_{p=1}^{\infty}$ is any countable collection of open and bounded sets that cover I, then it has a finite subcover $\{I_p\}_{p=1}^{n}$ for I (Heine-Borel). It follows then from the previous paragraph that

$$\ell(I) < \sum_{p=1}^{n} \ell(I_p) \leq \sum_{j=1}^{\infty} \ell(I_j)$$

This shows that $m^*(I) \geq \ell(I)$ and therefore $m^*(I) = \ell(I)$ in this case.

□

Proof.

• *Case I bounded interval.* Let a and b be the left and right endpoints of I. For $\epsilon > 0$ (small) let

$$J_1 = [a + \frac{\epsilon}{2}, \ b - \frac{\epsilon}{2}] \text{ and } J_2 = [a - \frac{\epsilon}{2}, \ b + \frac{\epsilon}{2}]$$

Hence J_1 and J_2 are closed bounded intervals with $\ell(J_1) = \ell(I) - \epsilon$, $\ell(J_2) = \ell(I) + \epsilon$ and $J_1 \subset I \subset J_2$. It follows from the first case (applied to J_1 and J_2) and the monotinicity of the outer measure that

$$\ell(I) - \epsilon = \ell(J_1) = m^* \ (J_1) \leq m^* \ (I) \leq m^* \ (J_2) = \ell(J_2) = \ell(I) + \epsilon$$

Therefore $m^* \ (I) = \ell(I)$ since $\epsilon > 0$ is arbitrary.

• *Case I unbounded interval.* If $I = \mathbb{R}$, then $m^* \ (I) = \infty = \ell(I)$. Suppose $I \neq \mathbb{R}$ and it is unbounded above so that $\ell(I) = \infty$. Let $c \in I$. For each $n \in \mathbb{N}$ the closed and bounded interval $I_n = [c, \ c + n]$ is contained in I. It follows from the monotonocity of the outer measure that

$$m^* \ (I) \geq m^* \ (I_n) = n \quad \text{for all } n \in \mathbb{N}$$

Therefore $m^* \ (I) = \infty = \ell(I)$. □

For set $E \subset \mathbb{R}$ and $s \in \mathbb{R}$, the translate of E is the set $E + s = \{x = e + s : \ e \in E\}$.
Note: Let $I \subset \mathbb{R}$ be an interval with endpoints a, b, then $I + s$ is an interval with endpoints $a + s, \ b + s$ and $\ell(I + s) = \ell(I)$.

Proposition

For every $E \subset \mathbb{R}$ and $s \in \mathbb{R}$, we have $m^* \ (E + s) = m^* \ (E)$. The outer measure m^* is invariant under translations.

Proof.

The collection of open and bounded intervals $\{I_j\}_{j=1}^{\infty}$ covers E if and only the collection of open and bounded translate intervals $\{I_j + s\}_{j=1}^{\infty}$ covers the translate $E + s$. This implies $m^* \ (E + s) = m^* \ (E)$. □

Proposition

Let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of subsets of $\mathbb{R}$. Then

$$m^* \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} m^* \left(E_j \right) . \qquad (*)$$

The outer measure m^* is subadditive.

Proof.

First note that if $\sum_j m^*(E_j) = \infty$, then $(*)$ holds trivially. Suppose then $\sum_j m^*(E_j) < \infty$. Let $\epsilon > 0$. It follows from the definition of m^* that for each j, there is a countable collection of open and bounded intervals $\{I_{j,k}\}_{k=1}^{\infty}$ such that $E_j \subset \bigcup_{k=1}^{\infty} I_{j,k}$ and

$$\sum_{k=1}^{\infty} \ell(I_{j,k}) < m^*(E_j) + \frac{\epsilon}{2^j} .$$

The collection of open and bounded intervals $\{I_{j,k}\}_{(j,k)\in\mathbb{N}\times\mathbb{N}}$ is countable and

$$\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} \left(\bigcup_{k=1}^{\infty} I_{j,k} \right) .$$

Furthermore,

$$\begin{aligned} m^*\left(\bigcup_{j=1}^{\infty} E_j\right) &\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \ell(I_{j,k}) \right) \\ &\leq \sum_{j=1}^{\infty} \left(m^*(E_j) + \frac{\epsilon}{2^j} \right) = \sum_{j=1}^{\infty} m^*(E_j) + \epsilon \end{aligned}$$

And $(*)$ follows since $\epsilon > 0$ is arbitrary. □

Measurable Sets

A set $E \subset \mathbb{R}$ is said to be measurable if for every set $S \subset \mathbb{R}$, the following equality holds:

$$m^*(S) = m^*(S \cap E) + m^*(S \cap E^c) \ , \qquad (**)$$

where $E^c = \mathbb{R} \backslash E$ is the $\mathbb{R}$-complement of E.

- $\emptyset$ and $\mathbb{R}$ are measurable.
- E is measurable if and only if E^c is measurable.
- If E is measurable and $T \subset \mathbb{R}$ is disjoint from E , then $m^*(E \cup T) = m^*(E) + m^*(T)$. Indeed $E \cup T = [(E \cup T) \cap E] \cup [(E \cup T) \cap E^c]$ and it follows from $(**)$ and from $E \cap T = \emptyset$ that
  $m^*(E \cup T) = m^*((E \cup T) \cap E) + m^*((E \cup T) \cap E^c) = m^*(E) + m^*(T)$.
- E is measurable if and only if for every set $S \subset \mathbb{R}$, $m^*(S) \geq m^*(S \cap E) + m^*(S \cap E^c)$. This is a consequence of the definition of measurable set and the subadditivity of the outer measure.
- Any set E with outer measure 0 ($m^*(E) = 0$) is measurable. Indeed, let $S \subset \mathbb{R}$. Then $S \cap E \subset E$ implies that $m^*(S \cap E) = 0$, and $S \cap E^c \subset S$ implies $m^*(S \cap E^c) \leq m^*(S)$. Hence, by the subadditivity we have

  $$m^*(S) \leq m^*(S \cap E) + m^*(S \cap E^c) \leq 0 + m^*(S)$$

  and $(**)$ holds.

Proposition

If E_1 and E_2 are measurable sets, then $E_1 \cup E_2$ is measurable. In general if $E_1, \cdots, E_n$ are measurable sets, then their union is also measurable.

Proof.

Let $S \subset \mathbb{R}$ be any set. The following identities (verification left as exercise) will be used

$$S \cap (E_1 \cup E_2) = (S \cap E_1) \cup \left(S \cap E_1^c \cap E_2\right) \; ; \; S \cap (E_1 \cup E_2)^c = \left(S \cap E_1^c\right) \cap E_2^c$$

Note that the subadditivity of m^* and the first identity above imply that

$$m^* \left(S \cap E_1\right) + m^* \left(S \cap E_1^c \cap E_2\right) \geq m^* \left(S \cap (E_1 \cup E_2)\right).$$

Using the measurability of E_1 and then of E_2 and the above identities, we have

$$\begin{aligned} m^* (S) &= m^* (S \cap E_1) + m^* \left(S \cap E_1^c\right) = m^* (S \cap E_1) + m^* \left(S \cap E_1^c \cap E_2\right) + m^* \left(S \cap E_1^c \cap E_2^c\right) \\ &\geq m^* (S \cap (E_1 \cup E_2)) + m^* \left(S \cap E_1^c \cap E_2^c\right) \\ &\geq m^* (S \cap (E_1 \cup E_2)) + m^* (S \cap (E_1 \cup E_2)^c) \geq m^* (S) \end{aligned}$$

Therefore $E_1 \cup E_2$ satisfies $(**)$ and it is measurable. The case of finite union of measurable sets follows by induction. □

Proposition

Let $E_1, \cdots, E_n$ be measurable sets that are mutually disjoint ($E_i \cap E_j = \emptyset$ if $i \neq j$) and let $S \subset \mathbb{R}$ be any set. Then

$$m^*\left(S \cap \left[\bigcup_{j=1}^{n} E_j\right]\right) = \sum_{j=1}^{n} m^*\left(S \cap E_j\right) \text{ and } m^*\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m^*\left(E_j\right)$$

Proof.

The second relation is a special case of the first with $S = \mathbb{R}$. We prove the first relation by induction. The case $n = 1$ is trivial. Suppose that the relation holds up to $n - 1$. Let $E_1, \cdots, E_n$ be measurable and mutually disjoint. Then for $S \subset \mathbb{R}$ we have the following identities

$$S \cap \left(\bigcup_{j=1}^{n} E_j\right) \cap E_n = S \cap E_n \text{ and } S \cap \left(\bigcup_{j=1}^{n} E_j\right) \cap E_n^c = S \cap \left(\bigcup_{j=1}^{n-1} E_j\right).$$

It follows from these identities, the measurability of E_n and the induction hypothesis that

$$\begin{aligned} m^*\left(S \cap \left[\bigcup_{j=1}^{n} E_j\right]\right) &= m^*\left(S \cap E_n\right) + m^*\left(S \cap \left(\bigcup_{j=1}^{n-1} E_j\right)\right) \\ &= m^*\left(S \cap E_n\right) + \sum_{j=1}^{n-1} m^*\left(S \cap E_j\right) \\ &= \sum_{j=1}^{n} m^*\left(S \cap E_j\right) \end{aligned}$$

□

Proposition

Let $\{E_j\}_{j=1}^{\infty}$ *be a countable collection of measurable sets. Then* $E = \bigcup_{j=1}^{\infty} E_j$ *is measurable.*

Proof.

First we prove that E can be expressed as the disjoint union of a countable family of measurable sets. For each $n \in \mathbb{N}$, let $E_n' = E_n \setminus \bigcup_{k=1}^{n-1} E_k = E_n \cap \left(\bigcup_{k=1}^{n-1} E_k \right)^c$. Then $E_n' \cap E_m' = \emptyset$ if $n \neq m$; E_n' is measurable (as a finite intersection of measurable sets); and $\bigcup_{n=1}^{\infty} E_n' = \bigcup_{n=1}^{\infty} E_n = E$.

For each $n \in \mathbb{N}$, let $F_n = \bigcup_{j=1}^{j} E_j'$. Then F_n is measurable (finite union of measurable sets). Note that $E^c \subset F_n^c$ for all n. Now let $S \subset \mathbb{R}$ be any set. Then we use the measurability of F_n and the monotonicity of m^* to get

$$m^* (S) = m^* (S \cap F_n) + m^* \left(S \cap F_n^c \right) \geq m^* (S \cap F_n) + m^* \left(S \cap E^c \right) .$$

Since F_n is the disjoint union of finitely many measurable set, then it follows from a previous proposition that $m^* (S \cap F_n) = \sum_{j=1}^{n} m^* \left(S \cap E_j' \right)$. Hence, $m^* (S) \geq \sum_{j=1}^{n} m^* \left(S \cap E_j' \right) + m^* \left(S \cap E^c \right)$ for all $n \in \mathbb{N}$, and it follows from the subadditivity of m^* that

$$m^* (S) \geq \sum_{j=1}^{\infty} m^* \left(S \cap E_j' \right) + m^* \left(S \cap E^c \right) \geq m^* (S \cap E) + m^* \left(S \cap E^c \right) .$$

Therefore E is measurable. □

Recall that a σ-algebra in $\mathbb{R}$ is a collection of subsets containing $\mathbb{R}$; is closed under formation of complement and countable union. The previous propositions imply that the collection of measurable sets forms a σ-algebra.

Theorem

An interval in $\mathbb{R}$ is measurable.

Proof.

Consider the case $I = (a,\ \infty)$ with $a \in \mathbb{R}$. Let $S \subset \mathbb{R}$ be any set. Since $m^*\ (S \backslash \{a\}) = m^*\ (S)$, we can assume $a \notin S$. Let $S^1 = S \cap I^c = S \cap (-\infty,\ a)$ and $S^2 = S \cap I = S \cap (a,\ \infty)$. To prove that I is measurable, it is enough to show that $m^*\ (S) \geq m^*\ \left(S^1\right) + m^*\ \left(S^2\right)$.

Let $\{I_j\}_{j=1}^{\infty}$ be a countable collection of open and bounded intervals that cover S. For each $j \in \mathbb{N}$ let $I_j^1 = I_j \cap (-\infty,\ a)$ and $I_j^2 = I_j \cap (a,\ \infty)$. Then I_j^1, I_j^2 are open bounded intervals such that $\ell(I_j) = \ell(I_j^1) + \ell(I_j^2)$ and the collections $\{I_j^1\}_{j=1}^{\infty}$ and $\{I_j^2\}_{j=1}^{\infty}$ cover S^1 and S^2, respectively. It follows from m^* as an infimum that $m^*\ \left(S^1\right) \leq \sum_{j=1}^{\infty} \ell(I_j^1)$ and $m^*\ \left(S^2\right) \leq \sum_{j=1}^{\infty} \ell(I_j^2)$. Hence,

$$m^*\ \left(S^1\right) + m^*\ \left(S^2\right) \leq \sum_{j=1}^{\infty} \left(\ell(I_j^1) + \ell(I_j^2)\right) \leq \sum_{j=1}^{\infty} \ell(I_j)\ .$$

Since $\{I_j\}_{j=1}^{\infty}$ is an arbitrary cover of S, it follows $m^*\ \left(S^1\right) + m^*\ \left(S^1\right) \leq m^*\ (S)$ and I is measurable.
Finally by using the fact that measurable sets form a σ-algebra, it can be proved that any interval is measurable. □

It follows from the preceding results that:

- Any open set is measurable (since it can be written as a countable union of intervals).
- Any closed set is measurable (as a complement of an open set)
- Any G_δ set is measurable (recall that a G_δ set is a set that can be written as the intersection of a countable collection of open sets)
- Any F_σ set is measurable (recall that an F_σ set is a set that can be written as the union of a countable collection of closed sets)
- Recall that a Borel σ-algebra is the σ-algebra generated by open sets (it is contained in any σ-algebra that contains open sets). Its members are called Borel sets. Since measurable sets form a σ-algebra, it follows that any Borel set is measurable.

We can summarize the above as:

Theorem

The collection $\mathcal{M}$ of measurable sets forms a σ-algebra that contains the σ-algebra of Borel sets. Each interval is measurable, each open set is measurable, each closed is measurable, each G_δ is measurable, and each F_σ is measurable.