Real Analysis MAA 6616
Lecture 6
Measurability: Inner and Outer Approximations

Lemma (1)

Let A be a measurable set in $\mathbb{R}$ with $m^*(A) < \infty$. Let $B \subset \mathbb{R}$ be any set containing A. Then $m^*(B \backslash A) = m^*(B) - m^*(A)$.

Proof.

Note that since $A \subset B$, then $B \cap A = A$ and $B \cap A^c = B \backslash A$. By definition of the measurability of A it follows that

$$m^*(B) = m^*(B \cap A) + m^*\left(B \cap A^c\right) = m^*(A) + m^*(B \backslash A) \ .$$

□

Theorem (2)

Let $A \subset \mathbb{R}$. The following properties are equivalent

1. *A is measurable;*
2. *For every $\epsilon > 0$, there exists an open set U, with $A \subset U$ such that $m^*(U \backslash A) < \epsilon$ (Approximation by open sets);*
3. *There exists a G_δ set G, with $A \subset G$ such that $m^*(G \backslash A) = 0$ (Approximation by G_δ sets);*
4. *For every $\epsilon > 0$, there exists a closed set V, with $V \subset A$ such that $m^*(A \backslash V) < \epsilon$ (Approximation by closed sets);*
5. *There exists an F_σ set F, with $F \subset A$ such that $m^*(A \backslash F) = 0$ (Approximation by F_σ sets);*

Proof.

"(1) $\Longrightarrow$ (2)" *Case :* $m^*(A) < \infty$. Let $\epsilon > 0$. It follows from the definition of m^* as a l.u.b that there a collection of open intervals $\{I_j\}_{j=1}^{\infty}$ that cover A and such that $\sum_{j=1}^{\infty} \ell(I_j) < m^*(A) + \epsilon$. Then $U = \bigcup_{j=1}^{\infty} I_j$ is an open set containing A such that $m^*(U) < \sum_{j=1}^{\infty} \ell(I_j) < m^*(A) + \epsilon$. Therefore (Lemma) $m^*(U \backslash A) = m^*(U) - m^*(A) < \epsilon$.

Case : $m^*(A) = \infty$. Let $\epsilon > 0$. For $m \in \mathbb{Z}$, let $J_m = [m, \ m+1)$ and $A_m = A \cap J_m$. Then A_m is measurable, $m^*(A_m) \leq 1$ for all m, and A is the disjoint union of the A_m's. It follows from the previous case that for each m there is collection of open intervals $\{I_{m,j}\}_{j=1}^{\infty}$ such that $A_m \subset U_m$ with $U_m = \bigcup_{j=1}^{\infty} I_{m,j}$,

$m^*(U_m \backslash A_m) = m^*(U_m) - m^*(A_m) < \frac{\epsilon}{2^{|m|}}$. The collection of open sets $\{U_m\}_{m\in\mathbb{Z}}$ cover A. Let $U = \bigcup_{m\in\mathbb{Z}} U_m$. Then U is an open set containing A and $m^*(U) \leq \sum_{m\in\mathbb{Z}} m^*(U_m) \leq \sum_{m\in\mathbb{Z}} \left(m^*(A_m) + \frac{\epsilon}{2^{|m|}} \right)$. Since

$U \backslash A = \bigcup_{m\in\mathbb{Z}} U_m \backslash A \ \subset \ \bigcup_{m\in\mathbb{Z}} (U_m \backslash A_m)$, then $m^*(U \backslash A) \leq \sum_{m\in\mathbb{Z}} m^*(U_m \backslash A_m) \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2^{|m|}} \leq 3\epsilon$.

"(2) $\Longrightarrow$ (3)" Suppose that A satisfies (2), then for each $j \in \mathbb{N}$, there is an open set U_j containing A such that $m^*\left(U_j \backslash A\right) < 1/j$. The G_δ set $G = \bigcap_{j=1}^{\infty} U_j$ contains A. Moreover $(G \backslash A) \subset (U_j \backslash A)$ for all j. Hence, $m^*(G \backslash A) < m^*\left(U_j \backslash A\right) < 1/j$ for all j. Therefore $m^*(G \backslash A) = 0$.

"(3) $\Longrightarrow$ (1)" Let G be a G_δ set containing A such that $m^*(G \backslash A) = 0$. Then G and $G \backslash A$ are measurable (as a G_δ-set and a set of measure 0). Then $A = G \cap (G \backslash A)^c$ is also measurable.

"(1) $\Longrightarrow$ (4)"Suppose A measurable, then A^c is measurable and so satisfies (2). Let $\epsilon > 0$. There exists an open set $U \supset A^c$ such that $m^*(U \backslash A^c) < \epsilon$. The set $V = U^c$ is closed and $V \subset A$. Since $A \backslash V = U \backslash A^c$, then $m^*(A \backslash V) < \epsilon$.

The the proofs of the remaining implications are left as exercises. □

The symmetric difference between the sets A and B is $A \triangle B = A \backslash B \cup B \backslash A$

Theorem (3)

Let $E \subset R$ be a measurable set with $m^*(E) < \infty$. Then for any $\epsilon > 0$, there exists a finite, disjoint, collection of open intervals $\{I_j\}_{j=1}^n$ such that $m^*\left(\left[\bigcup_{j=1}^n I_j\right] \triangle E\right) < \epsilon$.

Proof.

Let $\epsilon > 0$. There exists an open set $U \supset E$ such that $m^*(U \backslash E) < \epsilon/2$ (Theorem 2). Since $m^*(E) < \infty$, then $m^*(U) < \infty$ (consequence of Lemma 1). Since U is open then there is a countable disjoint collection of open intervals $\{I_j\}_{j=1}^\infty$ such that $U = \bigcup_{j=1}^\infty I_j$. Hence for any $m \in \mathbb{N}$, we have

$$\sum_{j=1}^m \ell(I_j) = m^*\left(\bigcup_{j=1}^m I_j\right) \leq m^*(U) < \infty .$$

Therefore $\sum_{j=1}^\infty \ell(I_j) < \infty$. We can therefore find $n \in \mathbb{N}$ such that $\sum_{j=n+1}^\infty \ell(I_j) < \epsilon/2$.

We have $\left[\bigcup_{j=1}^m I_j\right] \backslash E \subset U \backslash E$ and so $m^*\left(\left[\bigcup_{j=1}^n I_j\right] \backslash E\right) \leq m^*(U \backslash E) < \epsilon/2$. Also $E \subset U$ implies $E \backslash \left[\bigcup_{j=1}^n I_j\right] \subset \bigcup_{j=n+1}^\infty I_j$ and then $m^*\left(E \backslash \left[\bigcup_{j=1}^n I_j\right]\right) \leq \sum_{j=n+1}^\infty \ell(I_j) < \epsilon/2$. Therefore

$$m^*\left(E \triangle \left[\bigcup_{j=1}^n I_j\right]\right) = m^*\left(E \backslash \left[\bigcup_{j=1}^n I_j\right]\right) + m^*\left(\left[\bigcup_{j=1}^n I_j\right] \backslash E\right) < \epsilon$$

Let $\mathcal{M}$ be the σ-algebra of measurable sets. The Lebesgue measure of a set $E \in \mathcal{M}$ is defined as $m(E) = m^*(E)$.

Proposition (4)

If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is disjoint, then $\bigcup_{j=1}^{\infty} E_j \subset \mathcal{M}$ and $m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j)$.

Proof.

We already know that a countable union of measurable sets is measurable; that the outer measure is subadditive; and that it is additive on finite collections of disjoint measurable sets (see Lecture 5). Then $m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j)$, and $m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j)$. Therefore

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \geq m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j)$$

for all n. This implies the proposition. □

Proposition 4 together with results of Lecture 5 establish the following.

Theorem (5)

The Lebesgue measure $m : \mathcal{M} \longrightarrow \overline{\mathbb{R}^+}$ satisfies the following properties

- *If I is an interval $m(I) = \ell(I)$.*
- *m is translation invariant: For every $E \in \mathcal{M}$ and $s \in \mathbb{R}$, $m(E + s) = m(E)$.*
- *m is countably additive: If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is disjoint, then $m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j)$.*

Recall that a countable family of sets $\{E_j\}_{j\in\mathbb{N}}$ is said to be ascending if $E_n \subset E_{n+1}$ and descending if $E_n \supset E_{n+1}$ for all n.

Theorem (6)

The Lebesgue measure is continuous in the following sense:

- *If* $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ *is ascending, then* $m\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} m(E_j)$;
- *If* $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ *is descending, then* $m\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} m(E_j)$;

Proof.

Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ be ascending. *Case:* $m(E_{j_0}) = \infty$ *for some* j_0. In this case $m(E_j) = \infty$ for all $j \geq j_0$. Therefore $\infty = m\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} m(E_j)$. *Case:* $m(E_j) < \infty$ *for all* j. Define the collection $\{V_j\}_{j=0}^{\infty} \subset \mathcal{M}$ by $V_j = E_j \backslash E_{j-1}$, where $E_0 = \emptyset$. Then $V_k \cap V_j = \emptyset$ if $k \neq j$ and $\bigcup_{j=1}^{\infty} V_j = \bigcup_{j=1}^{\infty} E_j$. By using the countable additivity of the measure m and Lemma 1, we have

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(V_j) = \sum_{j=1}^{\infty} \left(m(E_j) - m(E_{j-1})\right) = \lim_{j\to\infty} m(E_j)$$

Now suppose that $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is descending. Define $\{W_j\}_{j=0}^{\infty} \subset \mathcal{M}$ by $W_j = E_1 \backslash E_j$ so that $W_j \subset W_{j+1}$ and $\bigcup_{j=1}^{\infty} W_j = E_1 \backslash \left(\bigcap_{j=1}^{\infty} E_j\right)$. By the ascending continuity property proved above and Lemma 1, the conclusion of the theorem follows from

$$m(E_1) - m\left(\bigcap_{j=1}^{\infty} E_j\right) = m\left(\bigcup_{j=1}^{\infty} W_j\right) = \lim_{j\to\infty} m(W_j) = m(E_1) - \lim_{j\to\infty} m(E_j).$$

□

Given a measurable set E a property $(\mathcal{P})$ is said to hold almost everywhere on E or that $(\mathcal{P})$ holds for almost all $x \in E$. If there exists a set $S \in \mathcal{M}$ of measure 0 such that $(\mathcal{P})$ holds for all $x \in E \backslash S$.

Theorem (7)

Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that $\sum_{j=1}^{\infty} m(E_j) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most a finite number of E_j's. More precisely, there exists a set of measure zero S such that for every $x \in \mathbb{R} \backslash S$ there exists $n \in \mathbb{N}$ such that $x \notin E_j$ for every $j \geq n$.

Proof.

Let $S = \limsup\{E_j\} = \bigcap_{n=1}^{\infty}\left(\bigcup_{j=n}^{\infty} E_j\right)$. For $x \in \mathbb{R}\backslash S$, there exists $n \in \mathbb{N}$ such that $x \notin \bigcup_{j=n}^{\infty} E_j$ and so $x \notin E_j$ for all $j \geq n$. We need to verify that $m(S) = 0$. For this we use the hypothesis $\sum_j m(E_j) < \infty$, the subadditivity and the continuity (Theorem 6) of the measure m to get

$$m(S) = \lim_{n \to \infty} m\left(\bigcup_{j=n}^{\infty} E_j\right) \leq \lim_{n \to \infty} \sum_{j=n}^{\infty} m(E_j) = 0.$$

□

Summary: Properties of m

- Monotonicity: If $A, B \in \mathcal{M}$ and $A \subset B$, then $m(A) \leq m(B)$
- Excision: If A, B are as above and $m(A) < \infty$, then $m(B \backslash A) = m(B) - m(A)$.
- Countable Additivity: If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is disjoint, then
$$m\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m(E_j)$$
- Countable Monotonicity: Let $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ and $A \in \mathcal{M}$ covered by $\{E_j\}_{j=1}^{\infty}$. Then $m(A) \leq \displaystyle\sum_{j=1}^{\infty} m(E_j)$.