Real Analysis MAA 6616
Lecture 7
Nonmeasurable Sets
The Cantor Set and
The Cantor-Lebesgue Function

Lemma (1)

Let $E \in \mathcal{M}$ be bounded. Suppose that there exists a bounded and countably infinite set $\Lambda \subset \mathbb{R}$ such that the collection $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint. Then $m(E) = 0$.

Proof.

Let M and K be positive numbers such that $E \subset [-M,\ M]$ and $\Lambda \subset [-K,\ K]$. We have $\lambda + E \subset [-(M+K),\ (M+K)]$ for all $\lambda \in \Lambda$. and $m\left(\bigcup_{\lambda \in \Lambda}(\lambda + E)\right) \leq 2(M+K)$. Since $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint, then it follows from the additivity of the measure m that

$$m\left(\bigcup_{\lambda \in \Lambda}(\lambda + E)\right) = \sum_{\lambda \in \Lambda} m(\lambda + E) = \sum_{\lambda \in \Lambda} m(E)\ .$$

The last sum would be ∞ if $m(E) > 0$. Therefore $m(E) = 0$. □

Let $E \subset \mathbb{R}$. Consider the rational equivalence relation define on E by $x \sim y$ if and only if $y - x \in \mathbb{Q}$. The set E is then decomposed into disjoint equivalence classes. Define the choice set $\mathcal{C}_E$ for this relation as a subset of E which consists of a single element from each equivalence class. Thus for every $s, t \in \mathcal{C}_E$, $s - t \notin \mathbb{Q}$ and for every $x \in E$ there exists a unique element $s \in \mathcal{C}_E$ such that $x - s \in \mathbb{Q}$. It follows that for every $\Lambda \subset \mathbb{Q}$, the collection $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ is disjoint.

Theorem (2-Vitali)

Let $E \subset \mathbb{R}$ with $m^*(E) > 0$. Then E contains a nonmeasurable set.

Proof.

First assume E is bounded. Let $M > 0$ such that $E \subset [-M, M]$. Let $\mathcal{C}_E$ be the choice set for the rational equivalence relation in E. We are going to show that $\mathcal{C}_E$ is not measurable.
By contradiction suppose that $\mathcal{C}_E$ is measurable. It follows from the property of the choice set that for every $\Lambda \subset \mathbb{Q}$ the collection $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ is disjoint. In particular when $\Lambda = \Lambda_0$ is countably infinite and bounded, we deduce from Lemma 1 that $m(\mathcal{C}_E) = 0$. Now consider $\Lambda_0 = \mathbb{Q} \cap [-2M, 2M]$. Then $E \subset \bigcup_{\lambda \in \Lambda_0}(\lambda + \mathcal{C}_E)$. Indeed, if $x \in E$, then there exists $c \in \mathcal{C}_E$ such that $\lambda = x - c \in \mathbb{Q}$. Moreover $|\lambda| \leq |x| + |c| < 2M$, so that $\lambda \in \Lambda_0$. It follows from the subadditivity of m^* that

$$m^*(E) \leq m\left[\bigcup_{\lambda \in \Lambda_0}(\lambda + \mathcal{C}_E)\right] = \sum_{\lambda \in \Lambda_0} m(\lambda + \mathcal{C}_E) = \sum_{\lambda \in \Lambda_0} m(\mathcal{C}_E) = 0\,.$$

This is contradiction since $m^*(E) > 0$. Therefore $\mathcal{C}_E$ is not measurable.
Next, suppose that E is unbounded. For every $n \in \mathbb{N}$, let $E_n = E \cap [-n, n]$. Then $E = \bigcup_{n \in \mathbb{N}} E_n$. Since, $m^*(E) > 0$, then there exists $N > 0$ such that $m^*(E_N) > 0$. The previous argument shows that the choice $\mathcal{C}_{E_N}$ (a subset of $E_N \subset E$) is not measurable. □

Theorem (3)

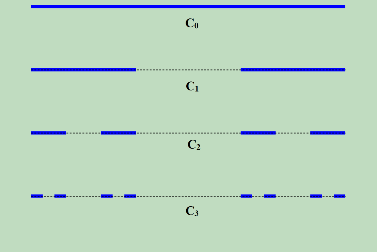
There exist disjoint sets A and B in $\mathbb{R}$ such that $m^*(A \cup B) < m^*(A) + m^*(B)$.

Proof.

Let E and C be arbitrary subsets of $\mathbb{R}$. Let $A = C \cap E$ and $B = C \cap E^c$. Then $A \cap B = \emptyset$. If the assertion of the theorem is not true, then $m^*(A \cup B) = m^*(A) + m^*(B)$. Since $A \cup B = C$ this means $m^*(C) = m^*(C \cap E) + m^*(C \cap E^c)$ for every C and E and the definition of measurability implies that all subsets of $\mathbb{R}$ are measurable which a contradiction. □

The Cantor Set: An Uncountable Set of Measure 0

Let $C_0 = [0,\ 1]$. Remove the middle third open interval $U_0 = (1/3,\ 2/3)$ from C_0 to obtain $C_1 = C_0 \backslash U_0$ as a union of two closed intervals $[0,\ 1/3]$ and $[2/3,\ 1]$ each of length $1/3$; From each component interval of C_1 remove the middle third open intervals $U_{1,1} = (1/9,\ 2/9)$ and $U_{1,2} = (7/9,\ 8/9)$ to obtain $C_2 = C_1 \backslash (U_{1,1} \cup U_{1,2})$ as a union of 2^2 closed intervals $[0,\ 1/9]$, $[2/9,\ 3/9]$, $[6/9,\ 7/9]$, $[8/9,\ 9/9]$, each of length $1/3^2$. Repeat this removal of "the middle third open intervals" so that at the n-th step we get a closed set $C_n \subset C_{n-1}$ as a union of 2^n closed intervals each with length $1/3^n$. The set $C = \bigcap_{n=1}^{\infty} C_n$ (which is not empty by the Nested-Set Theorem) is called the Cantor set.



Proposition (4)

The Cantor set C is uncountable and $m(C) = 0$.

Proof.

Since C is a countable intersection of closed sets, then it is measurable. Furthermore since $C \subset C_n$ and C_n is the disjoint union of 2^n intervals of length $1/3^n$, then $m(C) \leq m(C_n) = \left(\frac{2}{3}\right)^n$ for all n. Hence $m(C) = 0$.

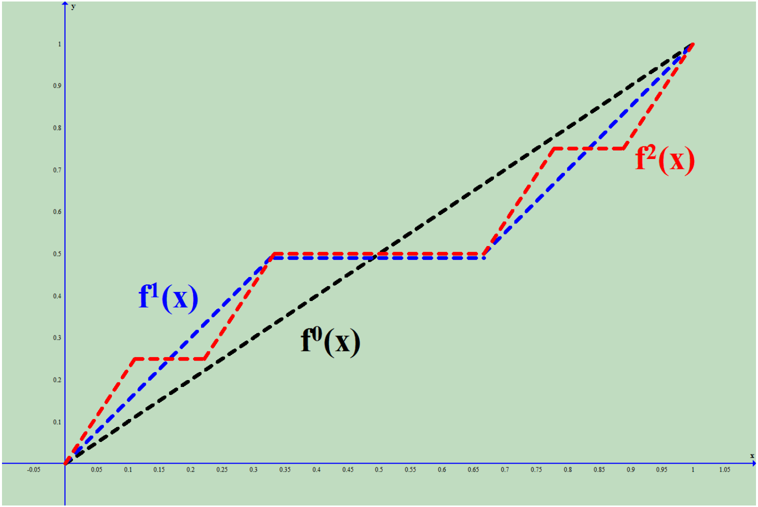
Now we show that C is uncountable. By contradiction, suppose that $C = \{c_n\}_{n=1}^{\infty}$ is countable. Since $c_1 \in C_1$ and C_1 is the disjoint union of two closed intervals. Let F_1 be the component of C_1 (on of the closed interval)that does not contain c_1. Note that $F_1 \cap C_2$ consists of two disjoint closed intervals each of length $1/3^2$. Let F_2 be one of the intervals of $F_1 \cap C_2$ that does not contain c_2. Hence $F_2 \subset F_1$ and $c_1, c_2 \notin F_2$. Suppose that we have constructed a descending family of closed intervals $F_1 \supset F_2 \supset \cdots \supset F_n$ such that $c_j \in F_j$ for $j = 1, \cdots, n$ and F_j a component of C_j with length 3^{-j}. The set $F_n \cap C_{n+1}$ consists of two disjoint closed interval of length $3^{-(j+1)}$. The point $c_{n+1} \in C_{n+1}$. Let F_{n+1} be one of the intervals of $F_n \cap C_{n+1}$ that does not contain c_{n+1}. We have then a countable collection of nested intervals $\{F_n\}_n$ with $F_n \subset C_n$. By the nested set Theorem $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let $c \in \bigcap_{n=1}^{\infty} F_n \subset C$. Since we assumed that C is countable, then there exists $m \in \mathbb{N}$ such that $c = c_m$ and this would mean that $c_m \in F_m$ which is contradiction. □

We construct an increasing piecewise linear continuous function on $[0,\ 1]$ with zero derivative almost everywhere.
First for a linear function $f(x) = mx + b$ we define the average on the interval $I = [a,\ b]$, as $f_{av}(I) = (f(a) + f(b))/2$. We use the nested collection $\{C_n\}_{n\in\mathbb{N}}$ to define the Cantor set set C and define a sequence of functions on $C_0 = [0,\ 1]$ as follows.

- $f^0(x) = x$.
- $f^1(x) = f^0_{av}(C_0)$ on the middle third interval $(1/3,\ 2/3) = C_0 \backslash C_1$ and $f^1(x)$ continuous on C_0 linear on each interval $[0,\ 1/3]$ and $[2/3,\ 1]$ such that $f^1(0) = 0$ and $f^1(1) = 1$.
- Suppose that $f^0, \cdots, f^n$ are defined on C_0. Define f^{n+1} as follows. Let $f^{n+1}(x) = f^n(x)$ on $C_0 \backslash C_n$. Let I be one of the 2^n closed intervals of length $1/3^n$ obtained at the n-th step in the construction of C. Let U be open middle third interval of I so that $I = J_1 \cup U \cup J_2$ where J_1 and J_2 are the closed intervals (of length $1/3^{n+1}$) contained in C_{n+1}. Define $f^{n+1}(x) = f^n_{av}(I)$ on U; f^{n+1} linear in each subinterval J_1, J_2 and such that f^{n+1} is continuous on I and $f^{n+1} = f^n$ at the extremities of I.

The sequence $\{f^n\}_n$ satisfies the following properties (proofs left as exercises)

- f^n is constant on each interval of $C_0 \backslash C_n$;
- f^n continuous and increasing on $[0,\ 1]$ with $f^n(0) = 0, f^n(1) = 1$


y
x
f¹(x)
f²(x)
f⁰(x)
-0.05
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Proposition (5)

For every $x \in [0,\ 1]$ and $n \in \mathbb{N}$ we have $\left|f^{n+1}(x) - f^n(x)\right| \leq \frac{1}{2^{n+1}}$. In particular the sequence $\{f^n\}$ converges uniformly on $[0,\ 1]$.

Proof.

Consider an interval $I = [a,\ b] = J_1 \cup U \cup J_2$, where U is the open middle third interval and J_1 and J_2 are remaining two closed intervals after removal of U. Consider a linear function $g(x) = 2\alpha x + \beta$ on I. Then the function $h(x)$ defined by $h(x) = g_{av}(I)$ continuous on I, linear on J_1 and J_2 and $h(a) = g(a), h(b) = g(b)$ is given by

$$h(x) = \begin{cases} 3\alpha x + \beta - \alpha a & \text{if } x \in [a,\ (2a+b)/3] \\ \alpha(a+b) + \beta & \text{if } x \in [(2a+b)/3,\ (a+2b)/3] \\ 3\alpha x + \beta - \alpha b & \text{if } x \in [(a+2b)/3,\ b] \end{cases}$$

A direct calculation shows that $|h(x) - g(x)| \leq |\alpha|\,(b-a)/3$.

We can use this observation to show (induction) that the slope of f^n in each interval of C_n is $\frac{3^n}{2^n}$. Since each interval of C_n has length $1/3^n$, then on each such interval we have.

$$\left|f^{n+1}(x) - f^n(x)\right| \leq \frac{3^n}{2^{n+1}} \cdot \frac{1}{3^n} = \frac{1}{2^{n+1}}$$

Since $f^{n+1} = f^n$ on $C_0 \setminus C_n$, the estimate follows on $[0,\ 1]$.
Finally, the uniform convergence follows directly from the above inequality. The sequence satisfies the uniform Cauchy criterion: For every $n,\ p \in \mathbb{N}$ and for every $x \in [0,\ 1]$ we have

$$\left|f^{n+p}(x) - f^n(x)\right| \leq \sum_{j=1}^{p} \left|f^{n+j}(x) - f^{n+j-1}(x)\right| \leq \sum_{j=1}^{p} \frac{1}{2^{n+j}} \leq \frac{1}{2^n}.$$

□

The limit function $\phi = \lim_{n \to \infty} f^n$ is called the Cantor-Lebesgue function on the interval I.

Theorem (6)

The Cantor-Lebesgue function ϕ is continuous and increasing on $[0,\ 1]$, maps $[0,\ 1]$ onto $[0,\ 1]$, it is differentiable on the open dense set $[0,\ 1] \backslash C$ and $\phi' \equiv 0$.

Proof.

The continuity of ϕ follows from the uniform convergence of the sequence $\{f^n\}$ and the monotonicity follows from the monotonicity of each f^n. Furthermore since $f^n([0,\ 1]) = [0,\ 1]$ for all n, then $\phi([0,\ 1]) = [0,\ 1]$. Since each f^n is constant on each interval contained on $[0,\ 1] \backslash C_n$, then ϕ is constant on each interval contained in open set $[0,\ 1] \backslash C$. and the conclusion follows. □

Now we use the Cantor-Lebesgue function ϕ to show that the image under a continuous function of a set of measure zero could be a set of positive measure and that the image of a measurable set could be a nonmeasurable set. For this consider the function ψ on $[0,\ 1]$ given $\psi(x) = \phi(x) + x$.

Proposition (7)

The function ψ satisfies the following properties:

1. $\psi : [0,\ 1] \longrightarrow [0,\ 2]$ *is an increasing homeomorphism.*
2. *Let C be the Cantor set. Then $\psi(C) \in \mathcal{M}$ and $m(\psi(C)) > 0$.*
3. *There exists a measurable set $E \subset C$ such that $\psi(E)$ is not measurable.*

Proof.

Since ϕ is increasing and the function x is strictly increasing, the function ψ is strictly increasing with $\psi(0) = 0$ and $\psi(1) = 2$ and as a sum of two continuous functions, ψ is continuous. It follows from the strict increase of ψ that it is bijective and $\psi^{-1} : [0,\ 2] \longrightarrow [0, 1]$ is also continuous (proof left as an exercise).
Consider the open set $U = [0,\ 1]\backslash C$. Then $[0,\ 1] = U \cup C$, a disjoint union. It follows from the strict monotonicity of ψ that $[0,\ 2] = \psi([0,\ 1]) = \psi(U) \cup \psi(C)$ and it follows from the fact that ψ is a homeomorphism that $\psi(U)$ is open and $\psi(C)$ is closed. Therefore both $\psi(U)$ and $\psi(C)$ are measurable. Let $\{I_n\}_n$ be the disjoint collection of all open middle third intervals removed in the construction of the Cantor set C. Then $U = \bigcup_{j=1}^{\infty} I_n$. Since $m(C) = 0$, then $m(U) = 1 = \sum_{n=1}^{\infty} \ell(I_n)$. Since the Cantor-Lebesgue function is constant on each I_n and since $\psi(x) = \phi(x) + x$, then for every n, $\psi(I_n)$ is an interval with $\ell(\psi(I_n)) = \ell(I_n)$. We have $\psi(U) = \bigcup_{n=1}^{\infty} \psi(I_n)$ a disjoint union. We deduce $m(\psi(U)) = \sum_{n=1}^{\infty} m(\psi(I_n)) = \sum_{n=1}^{\infty} m(I_n) = 1$. This means that since $[0,\ 2] = \psi(U) \cup \psi(C)$, we have $m(\psi(C)) = 2 - m(\psi(W)) = 1$.

To prove the third point, let E be a nonmeasurable set in $\psi(C)$. Such a nonmeasurable set exists since $m(\psi(C)) > 0$ (Vitali's Theorem). The set $A = \psi^{-1}(E) \subset C$ is measurable (as a subset of a set of measure 0). Therefore $\psi(A) = E$ is not measurable. □