Real Analysis MAA 6616
Lecture 8
Measurability in $\mathbb{R}^n$

$\mathbb{R}^p$ denotes the p-dimensional Euclidean space

$\mathbb{R}^p = \{x = (x_1, \cdots, x_p) : x_j \in \mathbb{R}, \ j = 1, \cdots, p\}$. The norm in $\mathbb{R}^p$ is $|x| = \sqrt{\sum_{j=1}^{p} x_j^2}$. The distance between points $x, \ y \ \in \mathbb{R}^p$ is $|y - x|$. The open ball with center x and radius r is $B_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$. If $I_1, \ \cdots \ I_p$ are intervals in $\mathbb{R}$, then $I = I_1 \times \cdots \times I_n$ is called an interval in $\mathbb{R}^n$.

A set $U \subset \mathbb{R}^p$ is said to be open if for every $x \in U$ there exists $r > 0$ such that the open ball $B_r(x)$ is contained in U ($B_r(x) \subset U$). A set $F \subset \mathbb{R}^p$ is closed if its complement $F^c = \mathbb{R}^p \backslash F$ is open. A set $C \in \mathbb{R}^p$ is said to compact if every open cover of C has a finite subcover. We have the following properties:

- The union of any collection of open sets in $\mathbb{R}^p$ is open and the union of a finite collection of closed sets in $\mathbb{R}^p$ is closed.
- The intersection of a finite collection of open sets in $\mathbb{R}^p$ is open and the intersection of any collection of closed sets in $\mathbb{R}^p$ is closed.
- A nonempty open set $U \subset \mathbb{R}^p$ can be written as a disjoint union of a countable collection of partly open intervals. A partly open interval in $\mathbb{R}^p$ is a set of the form $[a_1, \ b_1) \times \cdots \times [a_p, \ b_p)$ where $[a_j, \ b_j)$ is a semi closed interval in $\mathbb{R}$.
- Heine-Borel Theorem: A set $E \subset \mathbb{R}^p$ is compact if and only E is closed and bounded.
- Nested-Set Theorem: Let $\{F_n\}_n$ is a countable collection of closed and bounded sets such that $F_{n+1} \subset F_n \in \mathbb{R}^p$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

A sequence $\{x_n\}_n \subset \mathbb{R}^p$ converges to a limit $x \in \mathbb{R}^p$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ (i.e. $|x_n - x| < \epsilon$) for every $n \geq N$. A sequence $\{x_n\}_n \subset \mathbb{R}^p$ is said to be Cauchy if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for every $n,\ m \geq N$.

- Bolzano-Weierstrass Theorem: A bounded sequence in $\mathbb{R}^p$ has a convergent subsequence.
- Cauchy criterion: A sequence $\{x_n\}_n \subset \mathbb{R}^p$ is convergent if and only if it is Cauchy.
- A set $E \subset \mathbb{R}^p$ is compact if and only if every sequence $\{x_n\} \subset E$ has a convergent subsequence in E.

A function $f : E \subset \mathbb{R}^p \longrightarrow \mathbb{R}$ is continuous if for every open set $U \subset \mathbb{R}$, there exists an open set $V \subset \mathbb{R}^p$ such that $f^{-1}(U) = E \cap V$. This is equivalent to following: For every $x \in E$, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in B_\delta(x) \cap E$ we have $|f(y) - f(x)| < \epsilon$. The function f is uniformly continuous on E if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in E$ such that $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.

- $f : E \subset \mathbb{R}^p \longrightarrow \mathbb{R}$ is continuous if and only if for every sequence $\{x_n\}_n \subset E$ that converges to a point $x \in E$ we have $\lim_{n\to\infty} f(x_n) = f(x)$.
- Let $C \subset \mathbb{R}^p$ be a compact set and $f : C \longrightarrow \mathbb{R}$ be a continuous function. Then f is bounded and uniformly continuous

Lebesgue measure in $\mathbb{R}^p$

Let $I = \Pi_{j=1}^{d}[a_j\ b_j]$ be an interval in $\mathbb{R}^p$. The volume of I is $v(I) = \Pi_{j=1}^{p}(b_j - a_j)$. Let $E \subset \mathbb{R}^p$ be any set. Define the outer measure of E as $m^*(E) = \inf\left\{\sum_{j=1}^{\infty} v(I_j) : E \subset \bigcup_{j=1}^{\infty} I_j\right\}$. The outer measure is invariant under translation and is countably subadditive:
$m^*\left(\bigcup E_n\right) \leq \sum_n m^*(E_n)$.
A set $E \subset \mathbb{R}^p$ is Lebesgue measurable or simply measurable if for every $S \subset \mathbb{R}^p$, we have

$$m^*(S) = m^*(S \cap E) + m^*(S \cap E^c)\ .$$

As in the case of $\mathbb{R}$, the Lebesgue measure in $\mathbb{R}^p$ satisfies the following: $\emptyset$ and $\mathbb{R}^p$ are measurable; E is measurable if and only if E^c is measurable; if E is measurable and $T \subset \mathbb{R}^p$ is disjoint from E , then $m^*(E \cup T) = m^*(E) + m^*(T)$; E is measurable if and only if for every set $S \subset \mathbb{R}^p$, $m^*(S) \geq m^*(S \cap E) + m^*(S \cap E^c)$; Any set E with outer measure 0 ($m^*(E) = 0$) is measurable; a countable union of measurable sets is measurable; Let $E_1, \cdots, E_n$ be measurable sets that are mutually disjoint and let $S \subset \mathbb{R}^p$ be any set. Then

$$m^*\left(S \cap \left[\bigcup_{j=1}^{n} E_j\right]\right) = \sum_{j=1}^{n} m^*(S \cap E_j) \quad \text{and} \quad m^*\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} m^*(E_j)$$

Let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of measurable sets. Then $E = \bigcup_{j=1}^{\infty} E_j$ is measurable.

Let $A \subset \mathbb{R}^p$. The following properties are equivalent

- A is measurable;
- For every $\epsilon > 0$, there exists an open set U, with $A \subset U$ such that $m^* (U \backslash A) < \epsilon$ (Approximation by open sets);
- There exists a G_δ set G, with $A \subset G$ such that $m^* (G \backslash A) = 0$ (Approximation by G_δ sets);
- For every $\epsilon > 0$, there exists a closed set V, with $V \subset A$ such that $m^* (A \backslash V) < \epsilon$ (Approximation by closed sets);
- There exists an F_σ set F, with $F \subset A$ such that $m^* (A \backslash F) = 0$ (Approximation by F_σ sets);

Let $\mathcal{M}$ be the σ-algebra of measurable sets in $\mathbb{R}^d$. The Lebesgue measure or simply measure of a set $E \in \mathcal{M}$ is defined as $m(E) = m^*(E)$.

- If $I \subset \mathbb{R}^p$ is an interval $m(I) = v(I)$.
- The Lebesgue measure m is translation invariant: For every $E \in \mathcal{M}$ and $s \in \mathbb{R}$, $m(E+s) = m(E)$.
- The Lebesgue measure m is countably additive: If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is disjoint, then $m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j)$.