Real Analysis MAA 6616
Lecture 9
Lebesgue Measurable Functions

All functions considered here will be $\overline{\mathbb{R}}$-valued, where $\overline{\mathbb{R}}$ is the extended real line $\mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$. For a function $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$, we will use the following abbreviation $\{f < c\}$ for $\{x \in E : f(x) < c\}$ where $c \in \overline{\mathbb{R}}$. Similar abbreviations will be used for $\{f > c\}$, $\{f \geq c\}$, $\{f \leq c\}$, and $\{f = c\}$.

Proposition (1)

Let $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$. Then the following properties are equivalent:

1. *For every $c \in \overline{\mathbb{R}}$, $\{f > c\}$ is measurable;*
2. *For every $c \in \overline{\mathbb{R}}$, $\{f \geq c\}$ is measurable;*
3. *For every $c \in \overline{\mathbb{R}}$, $\{f < c\}$ is measurable;*
4. *For every $c \in \overline{\mathbb{R}}$, $\{f \leq c\}$ is measurable.*

Moreover, each one of these properties implies that $\{f = c\}$ is measurable.

Proof.

This proposition follows from the fact that the collection $\mathcal{M}$ of measurable sets is a σ-algebra: $(1) \iff (4)$ and $(2) \iff (3)$ follow from $\{f \leq c\} = \overline{\mathbb{R}} \backslash \{f > c\}$ and $\{f \geq c\} = \overline{\mathbb{R}} \backslash \{f < c\}$; $(1) \implies (2)$ and $(2) \implies (1)$ follow from $\{f \geq c\} = \bigcap_{n=1}^{\infty} \{f > c - \frac{1}{n}\}$ and $\{f > c\} = \bigcup_{n=1}^{\infty} \{f \geq c + \frac{1}{n}\}$.

Now assume that anyone of the four property holds (and so all four hold). Let $c \in \mathbb{R}$. Then $\{f = c\} = \{f \geq c\} \cap \{f \leq c\}$ is measurable. Also $\{f = -\infty\} = \bigcap_{n=1}^{\infty} \{f < -n\}$ and $\{f = \infty\} = \bigcap_{n=1}^{\infty} \{f > n\}$ are measurable. □

A function $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ defined on the measurable set E is said to be Lebesgue measurable if it satisfies any one of the properties of Proposition 1.

Theorem (2)

Let $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ with E measurable. Then f is measurable if and only if for every open set $U \subset \mathbb{R}$, the pre-image $f^{-1}(U) = \{x \in E : f(x) \in U\}$ is a measurable set in $\mathbb{R}^p$.

Proof.

"$\Longleftarrow$" Suppose that for every open set $U \subset \mathbb{R}, f^{-1}(U)$ is measurable. Let $c \in \mathbb{R}$. Since $(c, \infty) \subset \mathbb{R}$ is open, then $\{f > c\}$ is measurable, then f is measurable.

"$\Longrightarrow$" Suppose that f is measurable (and so it satisfies all four properties of Proposition 1). Let $U \subset \mathbb{R}$ be an open set. Then there is a countable collection of open and bounded intervals $\{I_j\}_{j\in\mathbb{N}} = \{(a_j, b_j)\}_{j\in\mathbb{N}}$ such that $U = \bigcup_{j\in\mathbb{N}}$. Note that $I_j = (a_j, \infty) \cap (-\infty, b_j)$ so that $f^{-1}(I_j) = \{f < b_j\} \cap \{f > a_j\}$ and $f^{-1}(I_j)$ is measurable. Therefore

$f^{-1}(U) = \bigcup_{j=1}^{\infty} f^{-1}(I_j)$ is measurable as a countable union of measurable sets. □

Theorem (3)

Let $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ be a continuous function and E measurable. Then f is measurable

Proof.

Let $U \subset \mathbb{R}$ be an open set. It suffices to show that $f^{-1}(U)$ is measurable (Theorem 2). Since f is continuous, then there exists an open set $V \subset \mathbb{R}^p$ such that $f^{-1}(U) = E \cap V$. Both E and V are measurable and so is their intersection $f^{-1}(U)$ □

Recall that a property $(\mathcal{P})$ is said to hold almost everywhere (abbreviated a.e) in a set E if there exists a set $Z \subset E$ of measure zero such that $(\mathcal{P})$ holds for every point $x \in E \backslash Z$.

Theorem (4)

Let $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ defined on the measurable set E.

1. *Suppose that f is measurable Let $g : E \longrightarrow \overline{\mathbb{R}}$. If $g = f$ a.e. on E, then g is measurable and for every $c \in \mathbb{R}$, $m(\{g > c\}) = m(\{f > c\})$*
2. *The function f is measurable on E if and only if for every measurable set $A \subset E$ the restrictions f_A and $f_{E\backslash A}$ of f to A and to $E\backslash A$ are measurable.*

Proof.

1. Let $Z = \{x \in E : \; g(x) \neq f(x)\}$ then $m(Z) = 0$. Let $c \in \mathbb{R}$ We need to show that $\{g > c\}$ is measurable. We have

$$\{g > c\} = \{x \in Z : g(x) > c\} \cup \{x \in E\backslash Z : f(x) > c\} = \{x \in Z : g(x) > c\} \cup (\{f > c\} \cap (E\backslash Z)) \; .$$

The set $\{g > c\}$ is measurable as an intersection of three measurable sets. Note that since $\{f > c\} = \{x \in E\backslash Z : f(x) > c\} \cup \{x \in Z : f(x) > c\}$ and $m(Z) = 0$, then $m(\{f > c\}) = m(\{x \in E\backslash Z : f(x) > c\})$. As a consequence we have $m(\{g > c\}) = m(\{f > c\})$.

2. Let $A \subset E$ measurable and $c \in \mathbb{R}$. We have

$$\{f > c\} = \{f_A > c\} \cup \{f_{E\backslash A} > c\} = (\{f > c\} \cap A) \cup (\{f > c\} \cap (E\backslash A)) \, .$$

It follows immediately that f is measurable if both f_A and $f_{E\backslash A}$ are measurable.

□

Theorem (5)

Let $f : E \subset \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ be finite a.e. in E and let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Assume that f is measurable. Then the composition $\alpha \circ f$ is measurable.

Proof.

Let $Z = \{f = \pm\infty\}$. Then Z has measure zero. Set $h = \alpha \circ f$ so that h is well defined on $E \backslash Z$ by $h(x) = \alpha(f(x))$. Let $U \subset \mathbb{R}$ be an open set. We need to verify that $h^{-1}(U) = f^{-1}(\alpha^{-1}(U)) \cap (E \backslash Z)$ is measurable. Since α is continuous, then $\alpha^{-1}(U)$ is open. Therefore, $f^{-1}(\alpha^{-1}(U))$ is measurable and so is $h^{-1}(U)$. □

Remark (1)

It follows from Theorem 5 that if f is measurable, then so are the functions $|f|, f^2, |f|^q$ for $q > 0$, $e^{\lambda f}$ etc.

Let $E \subset \mathbb{R}^p$. The characteristic function of E is the function

$$\chi_E : \mathbb{R}^p \longrightarrow \mathbb{R}; \quad \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

It follows directly from the definition of measurable functions that χ_E is measurable if and only if E is measurable.

Remark (2)

Composition of measurable functions is not necessarily measurable. Consider the $\psi(x) = x + \phi(x)$ where ϕ is the Cantor-Lebesgue function. We know that ψ is a homeomorphism between $[0,\ 1]$ and $[0,\ 2]$ and that there exists a set $S \subset C$ such $\psi(S)$ is not measurable (where C is the Cantor set). We can extend ψ as a global homeomorphism $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$ (for example $\Psi(x) = 2x$ for $x < 0$ and for $x > 1$). Hence Ψ^{-1} is continuous. Then the composition $h = \chi_S \circ \Psi^{-1}$ is not measurable. Indeed, $\{h = 1\} = \Psi(\chi_S(1)) = \psi(S)$ is not measurable.

Remark (3)

Given a function h, its measurability is not affected if the values are changed on a set of measure zero. With this understanding, the sum $f + g$ of two functions valued on $[-\infty,\ \infty]$ is well defined provided that the set of indeterminacy where $f(x) = \infty$ and $g(x) = -\infty$ has measure zero. Similarly for the product fg when the product takes the form $0 \cdot \infty$.

Theorem (6)

Let $f,\ g : E \longrightarrow \overline{\mathbb{R}}$ be measurable functions that are finite a.e. in E. Then

1. *The sum $f + g$ and difference $f - g$ are measurable.*
2. *The product fg is measurable.*
3. *The quotient f/g is measurable provided that $g \neq 0$ a.e. in E.*

Proof.

1. First observe that the set $\{f > g\} = \{x \in E : f(x) > g(x)\}$ is measurable. Indeed it can be written as a countable union of measurable sets: $\{f > g\} = \cup_{r \in \mathbb{Q}} (\{f > r\} \cap \{g < r\})$. Next, for any $\lambda \in \mathbb{R}$, the function $f + \lambda$ or $-f + \lambda$ is measurable since $\{f + \lambda > c\} = \{f > c - \lambda\}$ is measurable. Now to prove that $f + g$ is measurable, let $c \in \mathbb{R}$, then $\{f + g > c\} = \{f > c - g\}$ is measurable by the above observations.
2. We have $4fg = (f+g)^2 - (f-g)^2$. It follows from part 1 that $f \pm g$ are measurable and so $(f \pm g)^2$ are also measurable (Theorem 4). Consequently, fg is measurable.
3. Since up to a set of measure zero, we have $\{(1/g) > c\} = \{g < (1/c)\}$, the measurability of g implies that of $1/g$ and so of the quotient f/g (part 2)

□

Let $f_1, \cdots, f_n$ be functions defined on the same domain $E \subset \mathbb{R}^p$. Define the functions $\max\{f_1, \cdots, f_n\}$ and $\min\{f_1, \cdots, f_n\}$ in E by

$$\max\{f_1, \cdots, f_n\}(x) = \max\{f_1(x), \cdots, f_n(x)\} \text{ and}$$
$$\min\{f_1, \cdots, f_n\}(x) = \min\{f_1(x), \cdots, f_n(x)\}.$$

Theorem (7)

Let $f_1, \cdots, f_n : E \longrightarrow \overline{\mathbb{R}}$ *be measurable functions. Then the functions* $\max\{f_1, \cdots, f_n\}$ *and* $\min\{f_1, \cdots, f_n\}$ *are also measurable.*

Proof.

Let $c \in \mathbb{R}$. We have

$$\{\max\{f_1, \cdots, f_n\} > c\} = \bigcup_{j=1}^{n} \{f_j > c\} \text{ and } \{\min\{f_1, \cdots, f_n\} > c\} = \bigcap_{j=1}^{n} \{f_j > c\}.$$

Thus $\{\max\{f_1, \cdots, f_n\} > c\}$ is measurable as a finite union of measurable sets and $\{\min\{f_1, \cdots, f_n\} > c\}$ is measurable as a finite intersection of measurable sets. □

For a function $f : E \longrightarrow \overline{\mathbb{R}}$ we associate the functions:

$$|f| = \max\{f, -f\}, \; f^+ = \max\{f, 0\}, \text{ and } f^- = \max\{-f, 0\}.$$

Note that

$$f = f^+ - f^-$$

a difference of two nonnegative functions. Also $|f| = f^+ + f^-$.