THE HEAT EQUATION

The main equations that we will be dealing with are the *heat equation*, the *wave equation*, and the *potential equation*. We use simple physical principles to show how these equations are derived. We start the discussion with the heat equation

1. Heat Conduction

Our aim is to construct a mathematical model that describes temperature distribution in a body via *heat conduction*. There are other forms of heat transfer such as *convection* and *radiation* that will not be considered here. Basically, heat conduction in a body is the exchange of heat from regions of higher temperatures into regions with lower temperatures. This exchange is done by a transfer of kinetic energy through molecular or atomic vibrations. The transfer does not occur at the same rate for all materials. The rate of transfer is high for some materials and low for others. This *thermal diffusivity* depends mainly on the atomic structure of the material.

To interpret this mathematically, we first need to recall the notion of flux (that you might have seen in multivariable calculus). Suppose that a certain physical quantity Q flows in a certain region of the 3-dimensional space $\mathbb{R}^3$. For example Q could represent a mass (think of flowing water), or could represent energy (think of heat), or an electric charge. The *flux* corresponding to Q is a vector-valued function $\vec{q}$ whose direction indicates the direction of flow of Q and whose magnitude $|\vec{q}|$ the rate of change of Q per unit of area per unit of time. If for example Q measures gallons of water, then the units for the flux could be gallons per meter^2 per minute.

One way to understand the relationship between Q and $\vec{q}$ is a as follows. Let $m_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ with the standard canonical basis of orthonormal vectors $\vec{i}$, $\vec{j}$, $\vec{k}$. Consider a small rectangular surface S_1 centered at m_0 and parallel to the yz-plane. So the unit vector $\vec{i}$ is normal to S_1. Assume that S_1 has side lengths Δz and Δy (see Figure 1.)

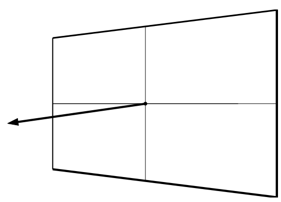

Figure 1. *Rectangular surface S_1, normal to the unit vector $\vec{i}$ and with vertices (y, z), $(y + \Delta y, z)$, $(y + \Delta y, z + \Delta z)$, and $(y, z + \Delta z)$.*

Let ΔQ denotes the net amount of the quantity Q that has crossed S_1 during the interval of time from t_0 to $t_0 + \Delta t$ (so $\Delta Q > 0$ if Q crosses S_1 in the direction

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of $\vec{i}$ and $\Delta Q < 0$ if Q crosses S_1 in the direction $-\vec{i}$.) Define

$$q_1(x_0, y_0, z_0, t_0) = \lim_{\substack{\Delta z \to 0 \\ \Delta y \to 0 \\ \Delta t \to 0}} \frac{\Delta Q}{\Delta y \Delta z \Delta t}.$$

Thus q_1 denotes the rate per unit of area per unit of time at the point m_0 at time t_0 at which Q crosses S_1 along the vector $\vec{i}$.

We can repeat the above construction for a vertical surface S_2 parallel to the xz-plane and centered at m_0 (thus normal to $\vec{j}$) and for a horizontal surface S_3 (normal to $\vec{k}$). We obtain in this way the rates $q_2(x_0, y_0, z_0, t_0)$ and $q_3(x_0, y_0, z_0, t_0)$. Set

$$\vec{q}(m_0, t_0) = q_1(m_0, t_0)\vec{i} + q_2(m_0, t_0)\vec{j} + q_3(m_0, t_0)\vec{k}.$$

Now, we redo the same construction at each point m of the region and for each time t under consideration. This gives the vector-valued function $\vec{q}(m, t)$. This is the flux of Q. For each time t, $\vec{q}$ is a vector field in a region of $\mathbb{R}^3$ (if Q flows in $\mathbb{R}^2$, then its flux would be valued in $\mathbb{R}^2$).

Note that if S is a surface containing a point m and has unit normal vector $\vec{n}$ at m, then the dot product $\vec{q}(m, t) \cdot \vec{n}$ denotes the rate of change of Q, per unit surface per unit of time, along the normal $\vec{n}$. The amount of Q that has crossed a small portion with area ΔS of S centered at m during an interval of time from t to $t + \Delta t$ is approximately

$$\Delta Q \approx (\vec{q}(m, t_0) \cdot \vec{n})\Delta S \Delta t.$$

Now some basics about heat transfer. As mentioned above, heat (a form of energy) can be viewed as a flowing quantity moving from regions of higher temperatures to regions of lower temperatures. This transfer is done according to the following:

- A change in the amount of heat in a body results in a change of its temperature;
- The heat flux is related to the gradient of the temperature; and
- The principle of energy conservation applies

In more details, the first point means that a change ΔQ in the amount of heat in a body of mass m results in a change Δu of its temperature. Furthermore, ΔQ is proportion al to the product $m\Delta u$. Hence, there is a constant c (that depends on the material structure of the body) such that

$$\Delta Q = c\, m\, \Delta u.$$

The constant c is called the *specific heat* of the material. It measures the amount of heat that is needed to raise the temperature of one unit of material by one degree. The units of c are

$$[c] = \text{(Energy)/(Mass)(Degree)=Joules/Kg.}^0\text{K} \quad \text{(in MKS System)}$$

The second point relates the heat flux $\vec{q}$ to the temperature gradient $\overrightarrow{\text{grad} u}$. It states that the heat flux is proportional to the temperature gradient. This law was discovered by J. Fourier in the 19th Century. So, if $\vec{q}(x, y, z, t)$ and $u(x, y, z, t)$ are, respectively, the heat flux and the temperature at a point (x, y, z) at time t, then there a positive real number K so that

$$\vec{q} = -K\overrightarrow{\text{grad}u} = -K(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}).$$

The constant K depends on the material of the body and is called the *thermal conductivity.* The presence of the minus (-) sign can be explained as follows. The direction of $\overrightarrow{\text{grad}u}$ indicates the direction along which u increases most rapidly, while $-\overrightarrow{\text{grad}u}$ indicates the direction in which u decreases most rapidly which must be the direction of the flow of heat. The units of K are

$$[K] = (\text{Energy}/(\text{Area})(\text{Time}))/(\text{Degree}/\text{Length})) = \text{Joules/m.s.}^0\text{K}$$

The third point (conservation of energy) means that the change in the amount of heat ΔQ during an interval of time of length Δt is equal to the amount of heat that has flown **into** the body minus the amount of heat that has flown **out** of the body during the interval of time.

$$\Delta Q = Q(t + \Delta t) - Q(t) = (\text{Amount INTO})\text{-}(\text{Amount OUT})$$

After dividing by Δt and letting $\Delta t \to 0$, we can express the above relation in terms of rates

$$\frac{dQ}{dt}(t) = (\text{Rate INTO})(t) - (\text{Rate OUT})(t).$$

2. The One Dimensional Heat Equation

Now we are ready to consider the problem of modeling the temperature distribution in a uniform thin rod of length L made of homogeneous material and with a constant cross section with area A.

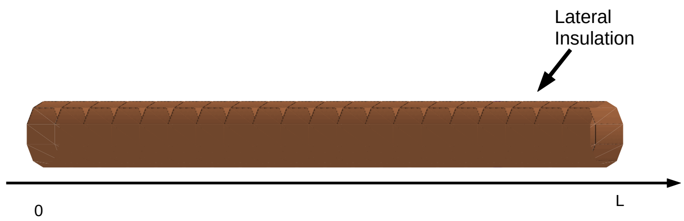

Figure 2. *Thin rod with lateral insulation*

We assume that the lateral surface of the rod is perfectly *insulated.* This means that there is NO heat transfer across the lateral surface of the rod. Thus heat flows only in the direction of the axis of the rod and there is no heat flow in any direction perpendicular to the axis of the rod. The temperature function u in the rod, at a given time t, is therefore the same at each point of a given cross section. But the temperature varies from one cross section to another. If the rod is set along the x-axis from 0 to L, then the temperature function depends on x with $0 \leq x \leq L$ and on time t:

$$u = u(x, t) \qquad 0 \leq x \leq L \quad t \geq 0.$$

The basic question is to understand how u varies with x and t.

We consider a small element of the rod between the cross sections at x and $x+\Delta x$ as in the Figure 3.

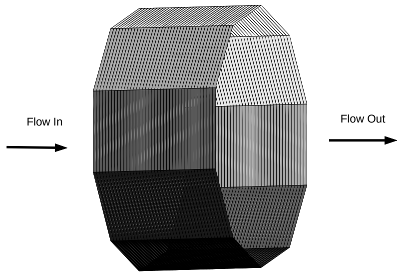

FIGURE 3. *A thin element of the rod between x and $x + \Delta x$*

To fix ideas, we assume that the left end of the element, at x, is warmer than the right end at $x + \Delta x$. The heat, then flows from left to right in the element. The heat flux $\vec{q}$ depends only x and t, is parallel to $\vec{i}$ (and so has no components along $\vec{j}$ and $\vec{k}$):

$$\vec{q}(x,t) = q(x,t)\vec{i}.$$

Then the rate at time t of heat flowing

- into the element is: $q(x,t)A$;
- out of the element is $q(x+\Delta x,t)A$

Let $Q(x,\Delta x,t)$ denotes the amount of heat in the element at time t. The principle of conservation of energy applied to the element is therefore

$$\frac{dQ}{dt}(x,\Delta x,t) = Aq(x,t) - Aq(x+\Delta x,t) = -A(q(x+\Delta x,t) - q(x,t)).$$

The law of thermodynamics (point 1) applied to the element of the rod is expressed as

$$Q(x,\Delta x,t) \approx c(\Delta m)u(x,t)\,,$$

where c is the specific heat of the rod and Δm is the mass of the element. We have used "$\approx$" instead of "$=$" because we made the assumption that u is approximately the same at each point of the element when Δx is small. Let ρ be the mass density of the rod (that we are assuming to be constant throughout the rod). Then

$$\Delta m = \text{Volume} \cdot \text{Density} = \rho A \Delta x.$$

The principle of conservation of energy can be rewritten as

$$c\rho A\Delta x \frac{\partial u}{\partial t} \approx -A(q(x+\Delta x,t) - q(x,t)).$$

Divide by $A\Delta x$ to obtain

$$c\rho \frac{\partial u}{\partial t} \approx -\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x}.$$

These approximations become better and better as Δx becomes smaller and smaller and at the limit ($\Delta x \to 0$), we obtain

$$c\rho \frac{\partial u}{\partial t}(x,t) = -\frac{\partial q}{\partial x}(x,t).$$

Now we apply Fourier's law (second point) to the element. It reads

$$q(x,t) = -K\frac{\partial u}{\partial x}(x,t)\,,$$

where K is the thermal conductivity of the rod (assumed to be constant). The above equation, take the form

$$c\rho\frac{\partial u}{\partial t}(x,t) = -\frac{\partial}{\partial x}\left(-K\frac{\partial u}{\partial x}\right)(x,t)\,,$$

or simply

$$\frac{\partial u}{\partial t}(x,t) = k\frac{\partial^2 u}{\partial x^2}(x,t)\,, \tag{1}$$

where we have set $k = K/c\rho$. Equation (1) is a *partial differential equation*, or simply PDE for short. This particular PDE is known as the *one-dimensional heat equation.* The constant k is the *thermal diffusivity* of the rod. The dimension of k is

$$[k] = \text{Area/Time}.$$

The higher the value of k is, the faster the material conducts heat.

3. The Two-Dimensional Heat Equation

Consider a thin homogeneous flat plate with a constant thickness h. Assume that the faces of the plate are perfectly insulated so that NO heat flows in the direction transversal to the plate. Hence heat is allowed to flow only in the directions of the plane of the plate. Suppose that the plate sits in the (x,y)-plane and denote its temperature function by u. Our assumption on the faces being insulated imply that u depends only on the location (x,y) and on time t:

$$u = u(x,y,t).$$

We are going to repeat the previous arguments to derive an equation for heat propagation in the plate. For this, consider a small rectangular element of the plate with vertices (x,y), $(x+\Delta x, y)$, $(x+\Delta x, y+\Delta y)$, and $(x, y+\Delta y)$. To fix ideas suppose that at time t, the bottom side of the rectangle is warmer than the top side and that the right side is warmer than the left side.

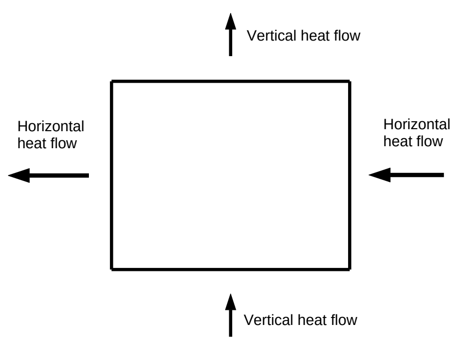

Figure 4. *A small rectangular element of the plate*

The amount of heat, at time t, in the small rectangle is

$$Q(x, y, \Delta x, \Delta y, t) \approx c(\Delta m)u(x, y, t)$$

where c is the specific heat and Δm is the mass of the rectangle. If ρ denotes the mass density, then $\Delta m = \rho h \Delta x \Delta y$. The rate of change of Q with respect to time is therefore

$$\frac{dQ}{dt} \approx ch\rho\Delta x\Delta y \frac{\partial u}{\partial t}. \tag{2}$$

Now we apply the principle of conservation of energy in the rectangle:

$$\frac{dQ}{dt} = \text{[Rate into] - [Rate out]}.$$

The heat flux $\vec{q}$ has two components a horizontal component q_1 and a vertical component q_2:

$$\vec{q}(x, y, t) = q_1(x, y, t)\vec{i} + q_2(x, y, t)\vec{j}.$$

In the figure we have $q_1 < 0$ and $q_2 > 0$. The rates of heat that flow into the rectangle through the vertical side (at $x + \Delta x$) and horizontal side (at y) are approximately

$$-q_1(x + \Delta x, y, t)h\Delta y \quad \text{and} \quad q_2(x, y, t)h\Delta x$$

(remember that h is the thickness of the plate). The rates of heat that flow out are approximately

$$-q_1(x, y, t)h\Delta y \quad \text{and} \quad q_2(x, y + \Delta y, t)h\Delta x.$$

It follows that the principle of conservation of energy can be expressed as

$$\frac{dQ}{dt} = -h\Delta y[q_1(x + \Delta x, y, t) - q_1(x, y, t)] - h\Delta x[q_2(x, y + \Delta y, t) - q_2(x, y, t)].$$

We replace dQ/dt by its expression given in (2), and after dividing by $h\Delta x\Delta y$, we obtain

$$c\rho\frac{\partial u}{\partial t}(x, t) \approx -\left[\frac{q_1(x + \delta x, y, t) - q_1(x, y, t)}{\Delta x} + \frac{q_2(x, y + \Delta y, t) - q_2(x, y, t)}{\Delta y}\right].$$

Again these approximations become better and better as Δx and Δy become smaller and smaller. At the limit, we obtain

$$c\rho\frac{\partial u}{\partial t}(x, t) = -\left(\frac{\partial q_1}{\partial x}(x, y, t) + \frac{\partial q_2}{\partial y}(x, y, t)\right). \tag{3}$$

Now Fourier's law ($\vec{q} = -K\overrightarrow{\text{grad}(u)}$) implies that

$$q_1(x, y, t) = -K\frac{\partial u}{\partial x}(x, y, t) \quad \text{and} \quad q_2(x, y, t) = -K\frac{\partial u}{\partial y}(x, y, t).$$

Therefore

$$\frac{\partial q_1}{\partial x} = -K\frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial q_2}{\partial y} = -K\frac{\partial^2 u}{\partial y^2}$$

(we are assuming that the conductivity K is constant). With these relation, expression (3) takes the form

$$\frac{\partial u}{\partial t}(x, y, t) = k\left(\frac{\partial^2 u}{\partial x^2}(x, y, t) + \frac{\partial^2 u}{\partial y^2}(x, y, t)\right) \tag{4}$$

where $k = K/c\rho$ is the diffusivity. Equation (4) is the *two-dimensional heat equation.*

4. Some Solutions Of The Heat Equation

The heat equation in one, two, or three space dimensions has infinitely many solutions. For the one dimensional heat equation $u_t = ku_{xx}$, the following functions

$$u_1(x,t) = \mathrm{e}^{-k\omega_1^2 t}\sin(\omega_1 x) \quad \text{and} \quad u_2(x,t) = \mathrm{e}^{-k\omega_2^2 t}\cos(\omega_2 x),$$

where ω_1 and ω_2 are arbitrary real numbers, are solutions (verify this claim as an exercise). From these solutions, we can built a new solution

$$v(x,t) = a_1 u_1(x,t) + a_2 u_2(x,t)$$

(verify that v is indeed a solution of the heat equation).

For the two-dimensional heat equation $u_t = k\Delta u$, the following functions are solutions

$$\mathrm{e}^{-k(\omega_1^2+\omega_2^2)t}\sin(\omega_1 x)\sin(\omega_2 y) \quad \text{or} \quad \mathrm{e}^{-k(\omega_1^2+\omega_2^2)t}\sin(\omega_1 x)\cos(\omega_2 y)$$

In the above we can change sin into cos and obtain again a solution. We can add two or more solutions, or multiply a solution by a constant and obtain again a solution (verify these claims).

5. Remarks

1. Consider a 3-dimensional body in the (x,y,z)-space. Suppose that heat is allowed to flow within the body in all three directions $\vec{i}$, $\vec{j}$, and $\vec{k}$. Now, the temperature function u depends effectively on the 3 space variables (x,y,z) and on time t. The above arguments carry over to establish the following equation

$$\frac{\partial u}{\partial t}(x,y,z,t) = k\left(\frac{\partial^2 u}{\partial x^2}(x,y,z,t) + \frac{\partial^2 u}{\partial y^2}(x,y,z,t) + \frac{\partial^2 u}{\partial z^2}(x,y,z,t)\right) \tag{5}$$

which is the *3-dimensional heat equation*.

2. The expression on the right of the heat equation, without the constant k in front of it, is known as the *Laplacian* of$u$. The Laplace operator, denoted Δ is

$$\begin{aligned}
\Delta &= \frac{\partial^2}{\partial x^2} && \text{in } \mathbb{R}; \\
\Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} && \text{in } \mathbb{R}^2; \\
\Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} && \text{in } \mathbb{R}^3
\end{aligned}$$

(Note that the Laplace operator Δ has nothing to do with the previously used notations Δx, Δt and so on that was used to denote a small increments in the variables x and t). The heat equation has then the form

$$\frac{\partial u}{\partial t} = k\Delta u\,. \tag{6}$$

3. Throughout the derivation of the heat equation, we have made a number of assumptions. In particular, we have assumed mass homogeneity of the body ($\rho =$ constant) and also homogeneity of the conductivity of heat ($K =$ constant). If these depend on (x,y,z), then the heat equation becomes

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right)$$

which after simplifications can be put in the form

$$\frac{\partial u}{\partial t} = k\Delta u + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma\frac{\partial u}{\partial z}$$

where all the coefficients k, α, β and γ might depend on the variables.

4. We have also assumed that there are no heat sources nor sinks within the body (a source emits heat and a sink absorbs heat). If there are such regions in the body, then for a uniform body, the heat equation becomes

$$\frac{\partial u}{\partial t} = k\Delta u + F(x, y, z, t)$$

where the function F accounts for the emission or absorbtion o heat.

6. Boundary Value Problems For The Heat Equation

Now we illustrate some typical boundary value problems (BVP for short) associated with the heat equation. We will only set up the problems. Their solvability has to wait until we develop the necessary tools.

6.1. **Example 1.** Suppose that a laterally insulated rod of length L has initial constant temperature of say 50^0. Then its left end $x = 0$ is immersed in a tank of icy water at 0^0 and its right end $x = L$ is immersed in a tank of boiling water at 100^0. We would like to set up a problem for the temperature function that would eventually enable us to find its (unique) solution

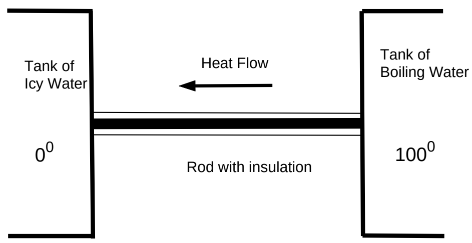

Figure 5. *A laterally insulated rod connecting two tanks with different temperatures*

Let $u(x, t)$ be the temperature at time t of the cross section through x of the rod. We know that u satisfies the PDE $u_t = ku_{xx}$, where k is the diffusivity of the rod. This equation has infinitely many solutions and by itself will not determine the (unique) solution of our specific problem. More information is needed. For this, we use the initial temperature distribution of the rod. That is for $t = 0$, the temperature in the rod is everywhere 50^0: $u(x, 0) = 50$ for every cross section through x. By immersing the ends in the (large) tanks, we are imposing that the ends of the rod have the the same temperature as those of the waters in the tanks. That is $u(0, t) = 0$ and $u(L, t) = 100$ for all time $t > 0$. All of this can be written formally as the following BVP

$$\begin{array}{lll} u_t(x,t) & = ku_{xx}(x,t) & \text{for } 0 < x < L, \quad t > 0; \\ u(x,0) & = 50 & \text{for } 0 < x < L; \\ u(0,t) & = 0 & \text{for } t > 0; \\ u(L,t) & = 100 & \text{for } t > 0; \end{array}$$

6.2. **Example 2.** Suppose that the rod in the previous example has an initial temperature (at time $t = 0$) given by a certain function $f(x)$ (for example $f(x)$ could be $x^2 - 3\sin(x)$). This time instead of immersing the ends, in the tanks, we insulate the ends of the rod. Hence, there no heat exchange between the rod and the outside. In particular, there is no heat flow through the ends. The heat flux

FIGURE 6. *A rod with a complete insulation*

$\vec{q}$ is 0 at $x = 0$ and at $x = L$ for each time $t > 0$. Remember that the heat flux is proportional to the temperature gradient (here u_x since there is only one space variable). Thus insulating the ends implies that $u_x(0, t) = 0$ and $u_x(L, t) = 0$. The BVP is therefore

$$\begin{array}{lll} u_t(x,t) = ku_{xx}(x,t) & 0 < x < L, & t > 0; \\ u(x,0) = f(x) & 0 < x < L; & \\ u_x(0,t) = 0 & t > 0; & \\ u_x(L,t) = 0 & t > 0. & \end{array}$$

Remark 1. We expect that the heat trapped inside the rod will distribute itself evenly throughout the rod until the temperature $u(x, t)$ approaches a constant value (the average of f) as t gets larger and larger.

6.3. **Example 3.** (A combination of the above situations). Suppose that the left end of the rod is insulated while the right end is immersed in the the tank of boiling water. Assume that the initial temperature of the rod is given by the function $f(x)$. The BVP is the following

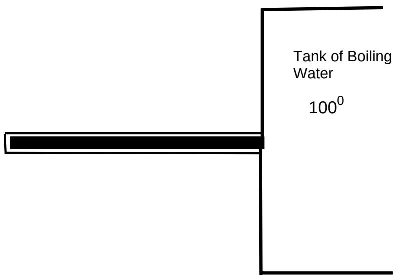

FIGURE 7. *A rod with one end insulated and the other kept at a constant temperature*

$$\begin{array}{lll} u_t(x,t) = ku_{xx}(x,t) & 0 < x < L, & t > 0; \\ u(x,0) = f(x) & 0 < x < L; & \\ u_x(0,t) = 0 & t > 0; & \\ u(L,t) = 100 & t > 0; & \end{array}$$

Remark 2. Other types of boundary conditions are possible. For example, if the end $x = 0$ has poor insulation, then the corresponding boundary condition might take the form

$$u_x(0,t) + au(0,t) = 0 \qquad t > 0$$

where a is a constant.

6.4. **Example 4.** Consider a thin plate in the shape of a quarter of a disk of radius R, whose faces (top and bottom) are insulated. Suppose that ($\imath$) initially (at time $t = 0$), the temperature distribution is given by the function $f(x, y)$; ($\imath\imath$) one radial edge is kept at constant temperature 0 while the other radial edge is insulated for all $t > 0$; and ($\imath\imath\imath$) the circular edge is kept at 100^0.

Insulating the vertical edge means that there is no heat exchange between the plate and the outside through the vertical side. This implies that the $\vec{\imath}$-th component of the heat flux $\vec{q}$ is 0 on the vertical edge. In terms of the temperature, it means $u_x = 0$ on the vertical edge. Thus the temperature function $u(x, y, t)$ satisfies the following

- the two-dimensional heat equation $u_t = \Delta u$;
- initial temperature $u(x, y, 0) = f(x, y)$;
- horizontal side $u(x, 0, t) = 0$;
- vertical side $u_x(0, y, t) = 0$;
- circular side $u(x, y, t) = 100$.

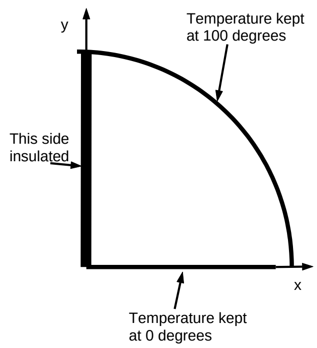

FIGURE 8. *The circular and horizontal sides of the plate are kept at constant temperature while the vertical side is insulated*

The BVP is then

$$\begin{array}{ll} u_t(x,y,t) = k\Delta u(x,y,t) & x^2+y^2<R^2,\ x>0,\ y>0, \quad t>0; \\ u(x,y,0) = f(x,y) & x^2+y^2<R^2\ x>0,\ y>0; \\ u(x,0,t) = 0 & 0<x<R, \quad t>0; \\ u(x,y,t) = 100 & x^2+y^2=R^2,\ x>0,\ y>0, \quad t>0; \\ u_x(0,y,t) = 0 & 0<y<R, \quad t>0. \end{array}$$

Note that it is more appropriate to set this problem in polar coordinates rather than in rectangular coordinates.

6.5. **Example 5.** If instead of having the vertical side insulated, it is the circular side that is insulated, then there will not be heat exchange across the circular side. This means that the heat flux $\vec{q}$ is perpendicular to the unit normal $\vec{n}$ to the circular side at each point of the circle. Thus $\vec{q} \cdot \vec{n} = 0$. In terms of the temperature u. It means that its normal derivative $\dfrac{\partial u}{\partial \vec{n}} = 0$ on the circle. Recall that the normal derivative is

$$\frac{\partial u}{\partial \vec{n}} = \mathrm{grad}(u) \cdot \vec{n}.$$

7. Exercises

In exercises 1 to 5, write the BVP for the temperature $u(x,t)$ in a homogeneous and laterally insulated rod of length L and diffusivity k in the following cases.

Exercise 1. The left end and right end are kept at temperature 0 degrees, the initial temperature at slice x is x degrees, $k=1$, and $L=20$.

Exercise 2. The left end is kept at temperature 10 degrees, the right end at temperature 50 degrees, the initial temperature at any slice x is 100 degrees, $k=2$, $L=50$.

Exercise 3. The left end is insulated, the right end at temperature 50 degrees, the initial temperature at any slice x is x^2 degrees, $k=1/2$, $L=50$.

Exercise 4. Both ends are insulated, the initial temperature at slice x is 100 degrees, $k=1$, $L=20$.

Exercise 5. The left end is controlled so that at time t, the temperature is $100\cos t$ degrees, the right end is insulated, the initial temperature is 50 degrees, $k=1$, $L=20$.

Exercise 6. Consider two identical, laterally insulated, uniform rods with diffusivity k and length L. These two rods are joined together to form a new rod of length $2L$ (the right end of rod 1 is joined to the left end of rod 2 as in the figure). Suppose

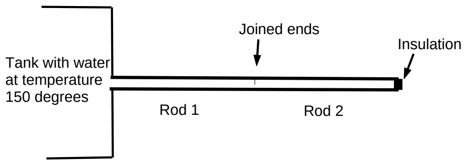

Figure 9. *Two rods joined to form a single rod*

that at time $t=0$ (just the moment when the rods are joined), the temperature of rod 1 is 0 degrees, that of rod 2 is 100 degrees, the right end of rod 2 in insulated and the left end of rod 1 is immersed in a tank with temperature of 150 degrees. Write the BVP for the temperature of the new rod of length $2L$.

Exercise 7. This time consider three identical, laterally insulated, uniform rods with diffusivity k and length L. These three rods are joined together to form a new rod of length $3L$ (the right end of rod 1 is joined to the left end of rod 2, and the right end of rod 2 is joined to the left end of rod 3 as in the figure). Suppose that

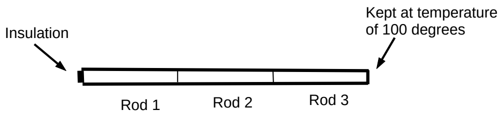

Figure 10. *Three rods joined to form a single rod*

the left end of rod 1 is insulated, the right end of rod 3 is kept at temperature of 100 degrees. Suppose that initially, the temperature in rod is 100 degrees, that in rod 2 is given by $100x(1 - x/L)$ (here x is the distance from the left end of rod 2 to the cross section), and the temperature in rod 3 is 0 degrees.

Exercise 8. The temperature function in a laterally insulated rod made of copper is found (say experimentally) to be $e^{-0.046t} \cos(0.2x)$. Find the thermal diffusivity k of copper.

Exercise 9. Verify that the function

$$u(x,t) = e^{-\pi^2 t/50} \sin \frac{\pi x}{10} - 5e^{-4\pi^2 t/50} \sin \frac{2\pi x}{10}$$

is a solution of the BVP

$$\begin{array}{ll} u_t = 2u_{xx} & 0 < x < 10, \quad t > 0 \\ u(0,t) = u(10,t) = 0 & t > 0 \\ u(x,0) = \sin \dfrac{\pi x}{10} - 5 \sin \dfrac{2\pi x}{10} & 0 < x < 10. \end{array}$$

If there is heat radiation within the rod of length L, then the 1-dimensional heat equation might take the form

$$u_t = ku_{xx} + F(x,t).$$

Exercise 10 to 13 deal with the steady-state situation. This means that the temperature u and F are independent on time t. In particular, $u_t \equiv 0$. The above heat equation becomes just an ordinary differential equation that you have learned how to solve in the first Differential Equation course (MAP3102).

Exercise 10. Find $u(x)$ if $F = 0$ (no radiation), $k = 3$, $u(0) = 2$, $u(L) = 10$.

Exercise 11. Find $u(x)$ if $F = 0$ (no radiation), $k = 1$, $u(0) = 2$, $u'(L) = 2$.

Exercise 12. Find $u(x)$ if $F(x) = x$, $k = 1$, $u(0) = 0$, $u(L) = 0$.

Exercise 13. Find $u(x)$ if $F(x) = \sin x$, $k = 2$, $u(0) = u'(0)$, $u(L) = 1$.

Exercises 13 to 17, deal with the temperature $u(x,y,t)$ in a homogeneous and thin plate. We assume that the top and bottom of the plate are insulated and the material has diffusivity k. Write the BVP in the following cases.

Exercise 13. The plate is a square with side length 1 with $k = 1$. The horizontal sides are kept at temperature 0 degrees, the vertical sides at temperature 100 degrees. The initial temperature in the plate is 50 degrees (constant throughout the plate).

Exercise 14. The plate is a 1×2 rectangle with $k = 2$. The vertical left side and the top horizontal sides are insulated. The vertical right and the horizontal bottom sides are kept at temperatures 0 and 100 degrees, respectively. The initial temperature distribution in the plate is $f(x,y) = \sin x \cos y$.

Exercise 15. The plate is triangular as in the figure. The vertical side is kept at temperature 0 degrees, the horizontal side at 50 degrees, and the slanted side is insulated. The initial temperature is 100 degrees throughout.

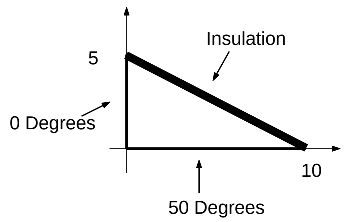

FIGURE 11. *Triangular plate*

Exercise 16. The plate is triangular as in the figure. The vertical side is kept at

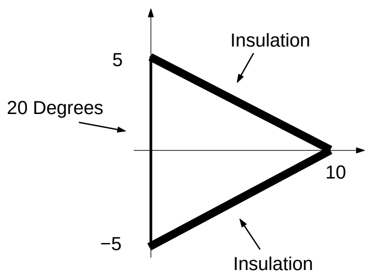

FIGURE 12. *Triangular plate*

temperature 20 degrees, the slanted sides are insulated and the initial temperature is given by the function xy.

Exercise 17. The plate is an angular segment of a circular ring as in the figure (the angle is $\pi/4$). The sides are kept as indicated in the figure and the initial

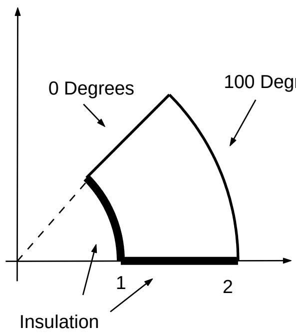

FIGURE 13. *Angular segment of a ring*

temperature in polar coordinates is $f(r, \theta) = r^2 \sin(4\theta)$.