NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS AND PROBLEMS IN HIGHER DIMENSIONS

We illustrate how eigenfunctions expansions can be used to solve more general boundary value problems. These include some nonhomogeneous problems and problems in higher dimensions.

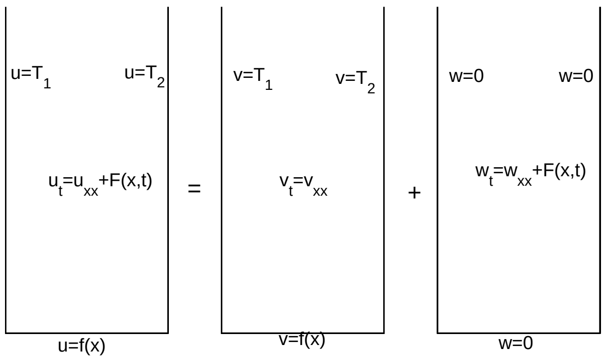
1. A heat propagation problem

Consider the problem

$$\begin{array}{ll} u_t = u_{xx} + F(x,t) & 0 < x < L,\ t > 0 \\ u(0,t) = T_1 & t > 0 \\ u(L,t) = T_2 & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{array}$$

This problem models heat propagation in a rod where the left end is kept at constant temperature T_1, the right end is kept at temperature T_2, the initial temperature is $f(x)$ and at each point x, there is heat radiating at the rate $F(x,t)$ at time t. Note that the PDE, the boundary conditions, and the initial condition, are nonhomogeneous.

We can use the principle of superposition to decompose this problem into two subproblems The subproblems for v and w are



$$\begin{cases} v_t = v_{xx} \\ v(0,t) = T_1 \\ v(L,t) = T_2 \\ v(x,0) = f(x) \end{cases} \quad \text{and} \quad \begin{cases} w_t = w_{xx} + F(x,t) \\ w(0,t) = 0 \\ w(L,t) = 0 \\ w(x,0) = 0 \end{cases}$$

The solution of the original problem is

$$u(x,t) = v(x,t) + w(x,t).$$

We solve separately, the v and the w problems.

The v-problem. To find the solution v, we can use the following steps:

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First, find the steady-state temperature $s(x)$ (independent on t) so that $s(0) = T_1$, $s(L) = T_2$ and $s''(x) = 0$. The function s is easily found

$$s(x) = (T_2 - T_1)\frac{x}{L} + T_1 \ .$$

Second, write a BVP for the function $y(x,t) = v(x,t) - s(x)$ (assuming that v solves the v-problem). We have

$$y_t = v_t - 0 = v_t \quad \text{and} \quad y_{xx} = v_{xx} - s''(x) = v_{xx} \ .$$

Hence $y(x,t)$ satisfies the heat equation $y_t = y_{xx}$. The boundary conditions for y are

$$\begin{aligned} y(0,t) &= v(0,t) - s(0) = T_1 - T_1 = 0 \\ y(L,t) &= v(L,t) - s(L) = T_2 - T_2 = 0 \end{aligned}$$

The initial condition for y is

$$y(x,0) = v(x,0) - s(x) = f(x) - (T_2 - T_1)\frac{x}{L} + T_1$$

The BVP problem for the function $y(x,t)$ is therefore the familiar problem

$$\begin{aligned} &y_t = y_{xx} \\ &y(0,t) = 0 \quad y(L,t) = 0 \\ &y(x,0) = f(x) - s(x) \end{aligned}$$

Third, This is a problem that we can solve by using separation of variables. We find

$$y(x,t) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\nu_n^2 t} \sin(\nu_n x) \ ,$$

where $\nu_n = n\pi/L$ and A_n is the n-th Fourier sine coefficient of the $f(x) - s(x)$:

$$A_n = \frac{2}{L}\int_0^L (f(x) - s(x)) \sin\frac{n\pi x}{L} dx$$

Conclude that the solution of the v-problem is

$$v(x,t) = y(x,t) + s(x) \ .$$

The w-problem. We indicate how to use eigenfunction expansion to construct a formal series solution. The eigenfunctions to be used are those of the associated SL-problem (in this case $X'' + \lambda X = 0$, $X(0) = X(L) = 0$):

$$\sin(\nu_n x), \qquad n \in \mathbb{Z}^+$$

First, expand the nonhomogeneous term $F(x,t)$ into a Fourier sine series in x (the variable t is considered as a parameter). We have,

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin(\nu_n x), \quad x \in (0,\ L), \quad t > 0 \ ,$$

where the n-th coefficient $F_n(t)$ depends on t and is given by

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t) \sin\frac{n\pi x}{L} dx \ .$$

Second, write the solution $w(x,t)$, has the Fourier sine series in x given by

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin\frac{n\pi x}{L} \ ,$$

where the Fourier coefficients $w_n(t)$ are function of t to be determined. Find ODEs for the coefficients $w_n(t)$. For this rewrite the PDE

$$w_t(x,t) - w_{xx}(x,t) = F(x,t)$$

by using the Fourier expansions for w and F. We obtain

$$\sum_{n=1}^{\infty} w_n'(t)\sin(\nu_n x) - \sum_{n=1}^{\infty} w_n(t)(-\nu_n^2)\sin(\nu_n x) = \sum_{n=1}^{\infty} F_n(t)\sin(\nu_n x)$$
$$\sum_{n=1}^{\infty} \left[w_n'(t) + \nu_n^2 w_n(t)\right]\sin(\nu_n x) = \sum_{n=1}^{\infty} F_n(t)\sin(\nu_n x)$$

After identifying the coefficients of $\sin(\nu_n x)$, we get the first order ODE for w_n

$$w_n'(t) + \nu_n^2 w_n(t) = F_n(t)\ .$$

Note that in addition to the ODE, w_n needs to satisfy the initial condition $w_n(0) = 0$ (this results from $w(x,0) = 0$).

Third, solve the ODE problem for w_n. This problem is of the form $y'(t)+ay(t) = g(t)$, $y(0) = 0$. Such an ODE can be solved by the method of variation of constant. The homogeneous equation $y' + ay = 0$ has general solution $y(t) = K\exp(-at)$. The general solution of the nonhomogeneous equation $y' + ay = g$ can be found by making K a function $K(t)$: $y(t) = K(t)\exp(-at)$. The function $K(t)$ satisfies $K'(t) = g(t)\exp(at)$. This together with the initial condition $y(0) = 0$ implies that $K(t) = \int_0^t g(s)\exp(as)ds$. The solution y is therefore

$$y(t) = \mathrm{e}^{-at}\int_0^t \mathrm{e}^{as} g(s)ds = \int_0^t \mathrm{e}^{-a(t-s)} g(s)ds\ .$$

Applying this to the coefficient w_n, we find that

$$w_n(t) = \mathrm{e}^{-\nu_n^2 t}\int_0^t \mathrm{e}^{\nu_n^2 s} F_n(s)ds.$$

By using the expression for F_n, we can also express w_n as

$$w_n(t) = \frac{2\mathrm{e}^{-\nu_n^2 t}}{L}\int_0^t\int_0^L \mathrm{e}^{\nu_n^2 s} F(x,s)\sin(\nu_n x)dxds$$

Example. Consider the BVP

$$\begin{array}{ll} u_t = u_{xx} + 100\sin^2 t & 0 < x < \pi,\ t > 0 \\ u(0,t) = 20 & t > 0 \\ u(\pi,t) = 100 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \end{array}$$

The v and w subproblems are

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = 20, \quad v(\pi,t) = 100 \\ v(x,0) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} w_t = w_{xx} + 100\sin^2 t \\ w(0,t) = 0 \quad w(\pi,t) = 0 \\ w(x,0) = 0 \end{cases}$$

The v-problem: First, find the steady-state temperature $s(x)$ with $s(0) = 20$ and $s(\pi) = 100$. We have

$$s(x) = \frac{80}{\pi}x + 20\ .$$

Let $y(x,t) = v(x,t) - s(x)$. The problem for $y(x,t)$ is:

$$y_t = y_{xx}, \quad y(0,t) = y(\pi,t) = 0, \quad y(x,0) = -s(x)$$

We find y by separation of variables and get

$$y(x,t) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-n^2 t} \sin(nx) \ ,$$

where

$$A_n = -\frac{2}{\pi}\int_0^{\pi} s(x)\sin(nx)dx = -\frac{2}{\pi}\int_0^{\pi}\left(\frac{80}{\pi}x + 20\right)\sin(nx)dx$$

Integration by parts gives

$$A_n = -40\frac{1-5(-1)^n}{\pi n} \ .$$

The solution of the v-problem: $v(x,t) = s(x) + y(x,t)$ is

$$v(x,t) = \frac{80}{\pi}x + 20 - \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1-5(-1)^n}{n}\mathrm{e}^{-n^2 t}\sin(nx)$$

The w-problem: We expand the nonhomogeneous term $100\sin^2 t$ in eigenfunctions $\sin(nx)$. That is, $100\sin^2 t = \sum_{n\geq 1} F_n(t)\sin(nx)$. We find

$$F_n(t) = \frac{200(1-(-1)^n)\sin^2 t}{n\pi} \ .$$

Thus,

$$100\sin^2 t = \sum_{j=0}^{\infty}\frac{400\sin^2 t}{(2j+1)\pi}\sin(2j+1)x, \quad x \in (0,\ \pi), \quad t > 0.$$

We expand the solution $w(x,t) = \sum_{n\geq 1} w_n(t)\sin(nx)$. The n-th coefficient $w_n(t)$ solves the ODE problem

$$w_n'(t) + n^2 w_n(t) = \frac{200(1-(-1)^n)}{n\pi}\sin^2 t, \qquad w_n(0) = 0.$$

The solution is

$$w_n(t) = \frac{200(1-(-1)^n)}{n\pi}\mathrm{e}^{-n^2 t}\int_0^t \mathrm{e}^{n^2 s}\sin^2 s\, ds \ .$$

To evaluate the last integral, we use the formula

$$\int \mathrm{e}^{as}\cos(bs)ds = \frac{\mathrm{e}^{as}(a\cos(bs) + b\sin(bs))}{a^2+b^2} + \text{Constant}$$

to obtain

$$\begin{aligned}\int_0^t \mathrm{e}^{n^2 s}\sin^2 s\, ds &= \frac{1}{2}\int_0^t \mathrm{e}^{n^2 s}(1-\cos(2s))ds = \frac{\mathrm{e}^{n^2 t}-1}{2n^2} - \frac{1}{2}\int_0^t \mathrm{e}^{n^2 s}\cos(2s)ds \\ &= \frac{\mathrm{e}^{n^2 t}-1}{2n^2} + \frac{n^2}{2(n^4+4)} - \frac{\mathrm{e}^{n^2 t}\left[n^2\cos(2t) + 2\sin(2t)\right]}{2(n^4+4)}\end{aligned}$$

Finally, the coefficient $w_{2j}(t) = 0$ and

$$w_{2j+1}(t) = \frac{200}{(2j+1)\pi}\left[\frac{1-\mathrm{e}^{-(2j+1)^2 t}}{(2j+1)^2} + \frac{(2j+1)^2\mathrm{e}^{-(2j+1)^2 t}}{(2j+1)^4+4} - \frac{(2j+1)^2\cos(2t) + 2\sin(2t)}{(2j+1)^4+4}\right]$$

Remark. Note that all the terms containing $e^{-\nu_n^2 t}$ converge to zero as $t \to \infty$. This means that such terms have negligible effect on the solution u. The long term behavior of the solutions are;

$$v(x,t) \approx \frac{80}{\pi}x + 20 \ ,$$

$$w_{2j+1}(t) \approx \frac{200}{(2j+1)\pi}\left[\frac{1}{(2j+1)^2} - \frac{(2j+1)^2\cos(2t) + 2\sin(2t)}{(2j+1)^4+4}\right]$$

the long term behavior of w is therefore

$$w(x,t) \approx \sum_{j=0}^{\infty} \frac{200}{(2j+1)\pi}\left[\frac{1}{(2j+1)^2} - \frac{(2j+1)^2\cos(2t) + 2\sin(2t)}{(2j+1)^4+4}\right]\sin(nx) \ .$$

2. Forced Vibrations of a String

The following boundary value problem models the vibrations of a string with an external force that depends on time. The endpoints are held fixed and the initial positions and velocity are zero.

$$\begin{array}{lll} & u_{tt} + 2au_t - c^2 u_{xx} = F(x,t) & 0 < x < L, \ t > 0 \\ (1) & u(0,t) = u(L,t) = 0 & t > 0 \\ & u(x,0) = 0, \ \ u_t(x,0) = 0 & 0 < x < L \end{array}$$

where $a \geq 0$ is the damping constant. We will assume that

$$F(x,t) = f(x)\cos(\omega_0 t) \ .$$

The eigenfunctions of the X-problem of the associated homogeneous problem are

$$X_n(x) = \sin(\nu_n x) \ , \quad \text{with} \quad \nu_n = \frac{n\pi}{L}, \ n \in \mathbb{Z}^+$$

We expand $F(x,t) = f(x)\cos(\omega_0 t)$ into a Fourier sine series (t is a parameter):

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t)\sin(\nu_n x) \ ,$$

where

$$F_n(t) = B_n\cos(\omega_0 t), \quad \text{with} \quad B_n = \frac{2}{L}\int_0^L f(x)\sin(\nu_n x)dx \ .$$

We are going to seek a formal series solution $u(x,t)$ in the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(\nu_n x) \ .$$

For each n, we write an ODE problem for the coefficient $C_n(t)$. We have

$$u_t = \sum_{n=1}^{\infty} C_n'(t)\sin(\nu_n x), \quad u_{tt} = \sum_{n=1}^{\infty} C_n''(t)\sin(\nu_n x), \quad u_{xx} = -\sum_{n=1}^{\infty} \nu_n^2 C_n(t)\sin(\nu_n x) \ .$$

The PDE for u becomes

$$\sum_{n=1}^{\infty}\left[C_n''(t) + 2aC_n'(t) + (c\nu_n)^2 C_n(t)\right]\sin(\nu_n x) = \sum_{n=1}^{\infty} F_n(t)\sin(\nu_n x) \ .$$

This means that $C_n(t)$ must satisfy the ODE

$$C_n''(t) + 2aC_n'(t) + (c\nu_n)^2 C_n(t) = F_n(t) = B_n\cos(\omega_0 t) \ .$$

The initial conditions $u(x,0) = 0$ and $u_t(x,0) = 0$ imply that $C_n(0) = 0$ and $C_n'(0) = 0$. Thus, for each $n \in \mathbb{Z}^+$, the function $C_n(t)$ must be a solution of the ODE problem

$$C_n''(t) + 2aC_n'(t) + (c\nu_n)^2 C_n(t) = B_n \cos(\omega_0 t)\,, \quad C_n(0) = C_n'(0) = 0\,.$$

This problem can be solved by the method of undetermined coefficients (UC-method). We are going to distinguish two cases depending on whether $a = 0$ or $a > 0$.

Forced vibrations without damping ($a = 0$). In this case the general solution of the ODE $y'' + (c\nu_n)^2 y = 0$ is

$$y(t) = K_1 \cos(c\nu_n t) + K_2 \sin(c\nu_n t).$$

To find a particular solution of the nonhomogeneous equation $y'' + (c\nu_n)^2 y = B_n \cos(\omega_0 t)$, we distinguish two cases: the case when $\cos(\omega_0 t)$ is not a solution of the homogeneous equation ($\omega_0 \neq c\nu_n$) and the case when $\cos(\omega_0 t)$ is a solution of the homogeneous equation ($\omega_0 = c\nu_n$).

Case $\omega_0 \neq c\nu_n$: A particular solution of the nonhomogeneous equation can be found in the form

$$y_p = P\cos(\omega_0 t) + Q\sin(\omega_0 t), \quad P,\ Q,\ \text{constants}\,.$$

The constants P and Q are found to be:

$$P = \frac{B_n}{(c\nu_n)^2 - \omega_0^2}\,, \qquad Q = 0\,.$$

The general solution of the ODE $y'' + (c\nu_n)^2 y = B_n \cos(\omega_0 t)$ is therefore

$$y(t) = \frac{B_n}{(c\nu_n)^2 - \omega_0^2}\cos(\omega_0 t) + K_1 \cos(c\nu_n t) + K_2 \sin(c\nu_n t).$$

In order for such a function to satisfy $y(0) = y'(0) = 0$, it is necessary to have

$$K_1 = -\frac{B_n}{(c\nu_n)^2 - \omega_0^2}, \qquad K_2 = 0\,.$$

Hence, in this case the coefficient $C_n(t)$ is:

$$C_n(t) = \frac{B_n}{(c\nu_n)^2 - \omega_0^2}\left(\cos(\omega_0 t) - \cos(c\nu_n t)\right)\,.$$

By using the trigonometric identity $\cos A - \cos B = 2\sin\dfrac{B+A}{2}\sin\dfrac{B-A}{2}$, we can express C_n in the form

$$C_n(t) = \frac{2B_n}{(c\nu_n)^2 - \omega_0^2}\sin\frac{(c\nu_n - \omega_0)t}{2}\sin\frac{(c\nu_n + \omega_0)t}{2}$$

Case $\omega_0 = c\nu_n$: A particular solution of the nonhomogeneous equation can be found in the form

$$y_p = Pt\cos(\omega_0 t) + Qt\sin(\omega_0 t), \quad P,\ Q,\ \text{constants}\,.$$

A calculation gives

$$P = 0, \qquad Q = \frac{B_n}{2\omega_0}.$$

The general solution of the nonhomogeneous ODE is

$$y(t) = \frac{B_n}{2\omega_0} t \sin(\omega_0 t) + K_1 \cos(\omega_0 t) + K_2 \sin(\omega_0 t)$$

In order for y to satisfy $y(0) = y'(0) = 0$, both constants K_1 and K_2 need to be 0. In this case the function $C_n(t)$ is

$$C_n(t) = \frac{B_n}{2\omega_0} t \sin(\omega_0 t) \ .$$

This case is known as *resonance* (the external frequency ω_0 is equal to one of the internal frequencies $c\nu_n$ of the string).

Summary of the case $a = 0$. The solution $u(x,t)$ of BVP (1) is given by the following

- Non-resonant case: $\omega_0 \neq c\nu_n \qquad \forall n \in Z^+$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2B_n}{(c\nu_n)^2 - \omega_0^2} \sin \frac{(c\nu_n - \omega_0)t}{2} \sin \frac{(c\nu_n + \omega_0)t}{2} \sin(\nu_n x) \ .$$

- Resonant case: $\exists j_0 \in \mathbb{Z}^+, \quad \omega_0 = c\nu_{j_0}$

$$\begin{aligned} u(x,t) \quad &= \frac{B_{j_0}}{2\omega_0} t \sin(\omega_0 t) \sin(\omega_0 x) + \\ &+ \sum_{n \neq j_0} \frac{2B_n}{(c\nu_n)^2 - \omega_0^2} \sin \frac{(c\nu_n - \omega_0)t}{2} \sin \frac{(c\nu_n + \omega_0)t}{2} \sin(\nu_n x) \end{aligned}$$

Forced vibrations with damping ($a > 0$). In this case the characteristic equation of the ODE $y'' + 2ay' + (c\nu_n)^2 y = 0$ is $m^2 + 2am + (c\nu_n)^2 = 0$ and the characteristic roots are

$$m_{1,2} = -a \pm \sqrt{a^2 - (c\nu_n)^2}.$$

For simplicity, let us assume that $0 < a < c\nu_1$ so that the quantity under the radical is negative (for all n). Set $\omega_n^2 = (c\nu_n)^2 - a^2$. The characteristic roots are then $m_{1,2} = -a \pm i\omega_n$. The general solution of the above homogeneous equation is

$$y(t) = \mathrm{e}^{-at} \left(K_1 \cos(\omega_n t) + K_2 \sin(\omega_n t) \right).$$

The nonhomogeneous equation

$$y'' + 2ay' + (c\nu_n)^2 y = B_n \cos(\omega_0 t)$$

has a particular solution of the form

$$y_p = P \cos(\omega_0 t) + Q \sin(\omega_0 t) \ .$$

The constants P and Q satisfy the system

$$\begin{array}{l} \left((c\nu_n)^2 - \omega_0^2\right) P + 2a\omega_0 Q = B_n \\ -2a\omega_0 P + \left((c\nu_n)^2 - \omega_0^2\right) Q = 0 \end{array}$$

Thus,

$$P = \frac{\left((c\nu_n)^2 - \omega_0^2\right) B_n}{\left((c\nu_n)^2 - \omega_0^2\right)^2 + (2a\omega_0)^2} \ , \qquad Q = \frac{2a\omega_0 B_n}{\left((c\nu_n)^2 - \omega_0^2\right)^2 + (2a\omega_0)^2} \ .$$

A particular solution of the ODE

$$C_n(t)'' + 2aC_n(t)' + (c\nu_n)^2 C_n(t) = B_n \cos(\omega_0 t)$$

is therefore

$$C_n(t) = \frac{B_n}{\left((c\nu_n)^2 - \omega_0^2\right)^2 + (2a\omega_0)^2} \left[\left((c\nu_n)^2 - \omega_0^2\right)\cos(\omega_0 t) + 2a\omega_0 \sin(\omega_0 t)\right] .$$

We have thus obtained a particular series solution of the nonhomogeneous wave equation

$$u_{tt} + 2au_t - c^2 u_{xx} = f(x)\cos(\omega_0 t)$$

as

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(\nu_n x)$$

where $C_n(t)$ is given by the above formula.

Example 1. Consider the BVP

$$\begin{array}{ll} u_{tt} - u_{xx} = \sin x \cos \dfrac{t}{2} & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & 0 < x < \pi \end{array}$$

In this situation $a = 0$, $\nu_n = n$ and $\omega_0 = 1/2$ (non resonant). The Fourier sine coefficients are $B_n = 0$ for $n \neq 1$ and $B_1 = 1$. The solution is therefore

$$u(x,t) = \frac{8}{3}\sin\frac{t}{4}\sin\frac{3t}{4}\sin x$$

Example 2. For the BVP

$$\begin{array}{ll} u_{tt} - u_{xx} = \cos(3t) & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 & 0 < x < \pi \end{array}$$

the external frequency $\omega_0 = 3$ is equal to one of the internal frequency $c\nu_3 = 3$ (here $c = 1$ and $\nu_n = n$). To find the series solution we need to expand $F(x,t) = \cos(3t)$ into Fourier sine series in x over the interval $[0,\ \pi]$ (which is just the Fourier sine series of 1 times $\cos(3t)$. We have

$$1 = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin(2j+1)x \qquad \forall x \in (0,\ \pi)\ .$$

In this case we have $B_{2j} = 0$, $B_{2j+1} = 4/(2j+1)\pi$. The solution u is

$$\begin{aligned} u(x,t) \quad &= \frac{B_3}{6} t\sin(3t)\sin(3x) + \\ &+ \sum_{j\neq 1}\frac{2B_{2j+1}}{(2j+1)^2 - 9}\sin\frac{(2j-2)t}{2}\sin\frac{(2j+4)t}{2}\sin((2j+1)x) \end{aligned}$$

Equivalently

$$u(x,t) = \frac{2t}{9\pi}\sin(3t)\sin(3x) + \frac{8}{\pi}\sum_{j\neq 1}\frac{\sin(j-1)t\ \sin(j+2)t}{(2j+1)\left[(2j+1)^2 - 9\right]}\sin((2j+1)x)$$

Example 3. Now we find a particular solution of the nonhomogeneous wave equation with damping

$$u_{tt} + 2u_t - u_{xx} = [\sin x - \sin(3x)]\cos(4t)$$

for $x \in [0,\ \pi]$ and $t > 0$. We have

$$a = 1,\ c = 1,\ \omega_0 = 4,\ \nu_n = n,\ \text{and}\ f(x) = \sin x - \sin(3x)\ .$$

Hence, $B_n = 0$ for $n \neq 1,\ 3,\ \ B_1 = 1$ and $B_3 = -1$. Thus the functions $C_n(t)$ are all zero except C_1 and C_3:

$$\begin{aligned} C_1(t) &= \frac{1}{(1-4^2)^2+8^2}\left[(1-4^2)\cos(4t)+8\sin(4t)\right] = \frac{-15\cos(4t)+8\sin(4t)}{289} \\ C_3(t) &= \frac{-1}{(3^2-4^2)^2+8^2}\left[(3^2-4^2)\cos(4t)+8\sin(4t)\right] = \frac{7\cos(4t)-8\sin(4t)}{113} \end{aligned}$$

A particular solution of the nonhomogeneous wave equation is therefore

$$u(x,t) = \frac{-15\cos(4t)+8\sin(4t)}{289}\sin x\ +\ \frac{7\cos(4t)-8\sin(4t)}{113}\sin(3x)$$

3. Double Fourier Series

Consider a function of two variables $f(x, y)$ with period $2L$ in x and with period $2H$ in y. That is,

$$f(x + 2L, y) = f(x, y + 2H) = f(x, y), \qquad \forall x, y\ .$$

We can associate to f a Fourier series in the variable x

$$f(x,y)\ \sim\ \frac{a_0(y)}{2} + \sum_{n\geq 1}\left(a_n(y)\cos\frac{n\pi x}{L} + b_n(y)\sin\frac{n\pi x}{L}\right) \tag{2}$$

with coefficients depending in the variable y:

$$\begin{aligned} a_n(y) &= \frac{1}{L}\int_{-L}^{L} f(x,y)\cos\frac{n\pi x}{L}dx, \qquad n = 0,\ 1, 2,\ \cdots \\ b_n(y) &= \frac{1}{L}\int_{-L}^{L} f(x,y)\sin\frac{n\pi x}{L}dx, \qquad n = 1, 2,\ \cdots \end{aligned}$$

Now each coefficient $a_n(y)$ and $b_n(y)$ is periodic in y and so we can associate to them Fourier series in the variable y:

$$\begin{aligned} a_n(y) &= \frac{a_{n0}}{2} + \sum_{m\geq 1}\left(a^1_{nm}\cos\frac{m\pi y}{H}) + a^2_{nm}\sin\frac{m\pi y}{H}\right) \\ b_n(y) &= \frac{b_{n0}}{2} + \sum_{m\geq 1}\left(b^1_{nm}\cos\frac{m\pi y}{H} + b^2_{nm}\sin\frac{m\pi y}{H}\right) \end{aligned}$$

If we substitute these expressions of a_n and b_n into (2) we obtain a double series called the *double Fourier series of* $f(x, y)$. More precisely we have the following

$$\begin{aligned} f(x,y)\ \ &\sim\ \frac{A_{00}}{4} + \frac{1}{2}\sum_{m\geq 1}\left(A_{0m}\cos\frac{m\pi y}{H} + B_{0m}\sin\frac{m\pi y}{H}\right) + \\ &+\frac{1}{2}\sum_{n\geq 1}\left(A_{n0}\cos\frac{n\pi x}{L} + C_{n0}\sin\frac{n\pi x}{L}\right) + \\ &\sum_{n\geq 1}\sum_{m\geq 1}\Big[A_{nm}\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H} + B_{nm}\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H} + \\ &\qquad + C_{nm}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H} + D_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\Big] \end{aligned}$$

The coefficients are given by

$$\begin{aligned} A_{nm} &= \frac{1}{LH}\int_{-H}^{H}\int_{-L}^{L} f(x,y)\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}dxdy \\ B_{nm} &= \frac{1}{LH}\int_{-H}^{H}\int_{-L}^{L} f(x,y)\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy \\ C_{nm} &= \frac{1}{LH}\int_{-H}^{H}\int_{-L}^{L} f(x,y)\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}dxdy \\ A_{nm} &= \frac{1}{LH}\int_{-H}^{H}\int_{-L}^{L} f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}dxdy \end{aligned}$$

The convergence results of Fourier series that we have seen in the case of a single variables have their counterparts in the case of multiple Fourier series. For example if $f(x,y)$ is smooth, then the association ($\sim$) is replaced by ($=$).

If we have a smooth function $F(x,y)$ defined over the rectangle $0 \le x \le L$, $0 \le y \le H$, then there are various ways in which we can represent $F(x,y)$ by a double trigonometric series: sine in x, sine in y; or sine in x, cosine in y, and so on. For example the Fourier cosine-sine series has the form

$$F(x,y) = \frac{1}{2}\sum_{m\ge1} B_{0m}\sin\frac{m\pi y}{H} + \sum_{n\ge1}\sum_{m\ge1} B_{nm}\sin\frac{m\pi y}{H}\cos\frac{n\pi x}{L}$$

where

$$B_{nm} = \frac{4}{LH}\int_0^L\int_0^H F(x,y)\sin\frac{m\pi y}{H}\cos\frac{n\pi x}{L}dydx\ .$$

Example. The expansion of the function $f(x,y) = 1$ over the square $[0,\ \pi]^2$ into a sine-sine trigonometric series is

$$1 = \sum_{n\ge1}\sum_{m\ge1} C_{nm}\sin(n\pi x)\sin(m\pi y)$$

where

$$C_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \sin(n\pi x)\sin(m\pi y)dydx = \frac{4\left[1-(-1)^n\right]\left[1-(-1)^m\right]}{\pi^2 mn}$$

We have then, for $(x,y) \in (0,\ \pi)^2$,

$$1 = \frac{16}{\pi^2}\sum_{j\ge0}\sum_{k\ge0}\frac{\sin(2j+1)x\ \sin(2k+1)y}{(2j+1)(2k+1)}$$

Remark. Multiple Fourier series in more than two variables can be defined in a similar way.

4. Application to Boundary Value Problems

We consider boundary value problems that can be solved by using multiple Fourier series.

4.1. **Vibrations of a rectangular membrane.** The following BVP models the vibrations of a rectangular membrane whose boundary is held fixed.

$$\begin{array}{ll} u_{tt} = c^2\left(u_{xx} + u_{yy}\right) & 0 < x < L,\ 0 < y < H,\ t > 0 \\ u(0,y,t) = u(L,y,t) = 0 & 0 < y < H,\ t > 0 \\ u(x,0,t) = u(x,H,t) = 0 & 0 < x < L,\ t > 0 \\ u(x,y,0) = f(x,y),\ \ u_t(x,y,0) = g(x,y) & 0 < x < L,\ 0 < y < H\ . \end{array}$$

The method of separation of variables for the homogeneous part leads to the following ODE problems

$$\left\{\begin{array}{l} X'' + \alpha X = 0 \\ X(0) = X(L) = 0 \end{array}\right. ,\quad \left\{\begin{array}{l} Y'' + \beta Y = 0 \\ Y(0) = Y(H) = 0 \end{array}\right. ,\quad T'' + c^2(\alpha+\beta)T = 0$$

where α and β are the separation constants. The eigenvalues of the X- problem and Y-problem are respectively

$$\begin{array}{llll} \alpha_n = \nu_n^2, & X_n(x)\sin(\nu_n x) & \nu_n = \dfrac{n\pi}{L}, & n \in \mathbb{Z}^+ \\ \beta_m = \mu_m^2, & Y_m(y)\sin(\mu_m y) & \mu_m = \dfrac{m\pi}{H}, & m \in \mathbb{Z}^+ \end{array}$$

set

$$\omega_{nm} = c\sqrt{\alpha_n + \beta_m} = c\sqrt{\nu_n^2 + \mu_m^2}\ .$$

For $n, m \in \mathbb{Z}^+$, the functions

$$\begin{array}{l} u^1_{nm}(x,y,t) = \cos(\omega_{nm}t)\sin(\nu_n x)\sin(\mu_m y) \\ u^2_{nm}(x,y,t) = \sin(\omega_{nm}t)\sin(\nu_n x)\sin(\mu_m y) \end{array}$$

The general series solution is

$$u(x,y,t) = \sum_{m\geq 1}\sum_{n\geq 1}\left[A_{nm}\cos(\omega_{nm}t) + B_{nm}\sin(\omega_{nm}t)\right]\sin(\nu_n x)\sin(\mu_m y)$$

In order for u to solve the nonhomogeneous conditions, we need to evaluate u and u_t at $t = 0$. First, we compute u_t

$$u_t(x,y,t) = \sum_{m\geq 1}\sum_{n\geq 1}\omega_{nm}\left[-A_{nm}\sin(\omega_{nm}t) + B_{nm}\cos(\omega_{nm}t)\right]\sin(\nu_n x)\sin(\mu_m y)$$

We have therefore

$$\begin{array}{l} u(x,y,0) = f(x,y) = \displaystyle\sum_{m\geq 1}\sum_{n\geq 1} A_{nm}\sin(\nu_n x)\sin(\mu_m y) \\ u_t(x,y,0) = g(x,y) = \displaystyle\sum_{m\geq 1}\sum_{n\geq 1}\omega_{nm}B_{nm}\sin(\nu_n x)\sin(\mu_m y) \end{array}$$

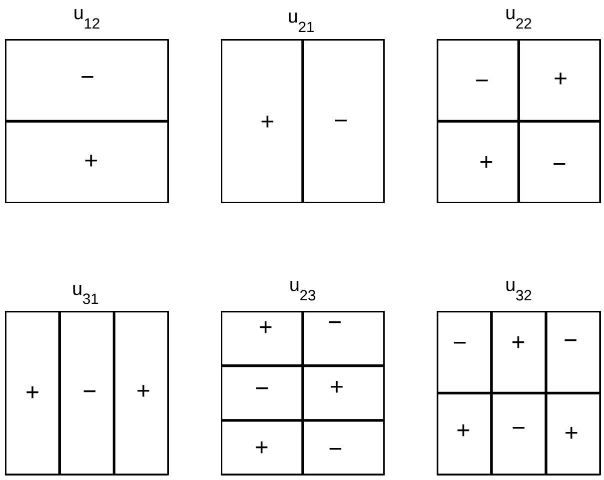
The above series are the sine-sine series of f and g. Thus

$$\begin{array}{l} A_{nm} = \dfrac{4}{LH}\displaystyle\int_0^L\int_0^H f(x,y)\sin(\mu_m y)\sin(\nu_n x)dydx\ , \\ B_{nm} = \dfrac{4}{\omega_{nm}LH}\displaystyle\int_0^L\int_0^H g(x,y)\sin(\mu_m y)\sin(\nu_n x)dydx\ . \end{array}$$

The solutions

$$\cos(\omega_{nm}t)\sin(\nu_n x)\sin(\mu_m y)$$

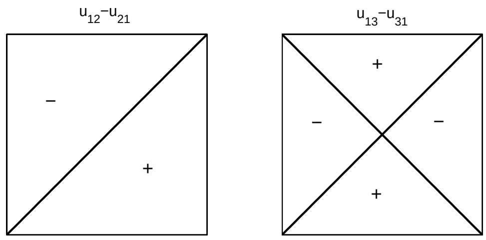
are called the (n,m)-*mode* of vibration of the membrane (also called *standing waves*). The function $\sin(\nu_n x)\sin(\mu_m y)$ is called the profile of the wave. The oscillations of the (n,m)-mode has frequency ω_{nm}. The set of points on the membrane that do not move are called the *nodal curves*. The nodal lines for the (n,m) mode is given by $\sin(\nu_n x)\sin(\mu_m y) = 0$. The following figure illustrates the nodal lines of some modes There are situations in which some different modes that have



the same frequency. For example when $L = H$, the two modes u_{12} and u_{21} have the same frequency

$$\omega_{12} = \omega_{21} = \frac{c\pi}{L}\sqrt{5}.$$

The mode $u_{12} - u_{21}$ is again a standing wave. The modes u_{13} and u_{31} have also the same frequency.



4.2. **Two-dimensional heat flow.** The heat flow in a rectangular plate with horizontal sides are kept at zero temperature and with vertical sides insulated lead to the following problem

$$\begin{array}{ll} u_t = k\,(u_{xx} + u_{yy}) & 0 < x < L,\ 0 < y < H,\ t > 0 \\ u_x(0,y,t) = u_x(L,y,t) = 0 & 0 < y < H,\ t > 0 \\ u(x,0,t) = u(x,H,t) = 0 & 0 < x < L,\ t > 0 \\ u(x,y,0) = f(x,y) & 0 < x < L,\ 0 < y < H\ . \end{array}$$

The separation of variables for the homogeneous part leads to the following ODE problems

$$\left\{\begin{array}{l} X'' + \alpha X = 0 \\ X'(0) = X'(L) = 0 \end{array}\right. , \quad \left\{\begin{array}{l} Y'' + \beta Y = 0 \\ Y(0) = Y(H) = 0 \end{array}\right. , \quad T' + k(\alpha+\beta)T = 0$$

where α and β are the separation constants. The eigenvalues and eigenfunctions of the X- problem

$$\alpha_0 = 0, \qquad X_0(x) = 1$$

and

$$\alpha_n = \nu_n^2, \qquad X_n(x)\cos(\nu_n x) \quad \nu_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}^+.$$

The eigenvalues and eigenfunctions of the Y-problem are

$$\beta_m = \mu_m^2, \quad Y_m(y)\sin(\mu_m y) \quad \mu_m = \frac{m\pi}{H}, \quad m \in \mathbb{Z}^+$$

Set

$$\lambda_{nm} = k(\alpha_n + \beta_m) = c(\nu_n^2 + \mu_m^2) \quad n = 0,\ 1,\ 2, \cdots, \quad m = 1,\ 2,\ 3, \cdots .$$

The solutions with separated variables of the homogeneous part of the problem are:

$$\begin{array}{l} u_{0m}(x,y,t) = \mathrm{e}^{-\lambda_{0m}t}\sin(\mu_m y) \quad m \in \mathbb{Z}^+ \\ u_{nm}(x,y,t) = \mathrm{e}^{\lambda_{nm}t}\cos(\nu_n x)\sin(\mu_m y) \quad n, m \in \mathbb{Z}^+ \end{array}$$

The series solution is

$$u(x,y,t) = \sum_{m\geq 1} A_{0m}\mathrm{e}^{-\lambda_{0m}t}\sin(\mu_m y) + \sum_{m\geq 1}\sum_{n\geq 1} A_{nm}\mathrm{e}^{\lambda_{nm}t}\cos(\nu_n x)\sin(\mu_m y)$$

The coefficients A_{nm} are obtained from the nonhomogeneous condition

$$u(x,y,0) = f(x,y) = \sum_{m\geq 1} A_{0m}\sin(\mu_m y) + \sum_{m\geq 1}\sum_{n\geq 1} A_{nm}\cos(\nu_n x)\sin(\mu_m y)\ .$$

This is the Fourier cosine-sine series of f in the rectangle. Thus

$$\begin{array}{l} A_{0m} = \dfrac{2}{LH}\displaystyle\int_0^L\int_0^H f(x,y)\sin(\mu_m y)dydx, \\ A_{nm} = \dfrac{4}{LH}\displaystyle\int_0^L\int_0^H f(x,y)\sin(\nu_n x)\sin(\mu_m y)dydx, \end{array}$$

4.3. **A Poisson problem in a cube.** We use triple Fourier series to solve the following *Poisson* problem in the cube $[0,\ \pi]^3$.

$$\begin{array}{ll} u_{xx} + u_{yy} + u_{zz} = F(x,y,z) & 0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi \\ u(0,y,z) = u(\pi,y,z) = 0 & 0 < y < \pi,\ 0 < z < \pi \\ u(x,0,z) = u(x,\pi,z) = 0 & 0 < x < \pi,\ 0 < z < \pi \\ u(x,y,0) = u(x,y,\pi) = 0 & 0 < x < \pi,\ 0 < y < \pi \end{array}$$

(such problem models an electric potential in the cube). The homogeneous boundary conditions suggest that we seek a series solution in the eigenfunctions of the SL problems.

$$\left\{\begin{array}{l} X'' + \alpha X = 0 \\ X(0) = X(\pi) = 0 \end{array}\right., \quad \left\{\begin{array}{l} Y'' + \beta Y = 0 \\ Y(0) = Y(\pi) = 0 \end{array}\right., \quad \left\{\begin{array}{l} Z'' + \gamma Y = 0 \\ Z(0) = Z(\pi) = 0 \end{array}\right.$$

The eigenvalues and eigenfunctions are respectively

$$\begin{array}{lll} \alpha_j = j^2, & X_j(x) = \sin(jx), & j \in \mathbb{Z}^+ \\ \beta_k = k^2, & Y_k(y) = \sin(ky), & k \in \mathbb{Z}^+ \\ \gamma_l = l^2, & Z_l(z) = \sin(lz), & l \in \mathbb{Z}^+ \end{array}$$

We seek a series solution is

$$u(x,y,z) = \sum_{j\geq 1}\sum_{k\geq 1}\sum_{l\geq 1} C_{jkl}\sin(jx)\sin(ky)\sin(lz)$$

Note that since

$$\Delta\left(\sin(jx)\sin(ky)\sin(lz)\right) = -(j^2+k^2+l^2)\sin(jx)\sin(ky)\sin(lz)\ ,$$

then

$$\Delta u = F = -\sum_{j\geq 1}\sum_{k\geq 1}\sum_{l\geq 1}(j^2+k^2+l^2)C_{jkl}\sin(jx)\sin(ky)\sin(lz)$$

The last series is the triple Fourier sine-sine-sine series of the nonhomogeneous term $F(x,y,z)$. The coefficients C_{jkl} are therefore

$$C_{jkl} = \frac{-8}{\pi^3(j^2+k^2+l^2)}\int_0^\pi\int_0^\pi\int_0^\pi F(x,y,z)\sin(jx)\sin(ky)\sin(lz)dxdydz\ .$$

5. Exercises.

In exercises 1 to 7 solve the nonhomogeneous boundary value problems

Exercise 1.

$$\begin{array}{ll} u_t = u_{xx} & 0<x<\pi,\ t>0 \\ u(0,t)=1,\ u(\pi,t)=3 & t>0 \\ u(x,0)=x & 0<x<\pi \end{array}$$

Exercise 2.

$$\begin{array}{ll} u_t = u_{xx} + \mathrm{e}^{-x} & 0<x<\pi,\ t>0 \\ u(0,t)=0,\ u(\pi,t)=0 & t>0 \\ u(x,0)=0 & 0<x<\pi \end{array}$$

Exercise 3.

$$\begin{array}{ll} u_t = u_{xx} - x & 0<x<\pi,\ t>0 \\ u(0,t)=0,\ u(\pi,t)=0 & t>0 \\ u(x,0)=x & 0<x<\pi \end{array}$$

Exercise 4.

$$\begin{array}{ll} u_t = u_{xx} + 2t & 0<x<\pi,\ t>0 \\ u(0,t)=0,\ u(\pi,t)=100 & t>0 \\ u(x,0)=0 & 0<x<\pi \end{array}$$

Exercise 5.

$$\begin{array}{ll} u_{tt} = u_{xx} - g & 0<x<\pi,\ t>0 \\ u(0,t)=0,\ u(\pi,t)=0 & t>0 \\ u(x,0)=0,\ u_t(x,0)=\sin x & 0<x<\pi \end{array}$$

where g is a constant (gravitational for example).

Exercise 6.

$$\begin{array}{ll} u_{tt} = u_{xx} + \sin(2x) & 0<x<\pi,\ t>0 \\ u(0,t)=0,\ u(\pi,t)=0 & t>0 \\ u(x,0)=\sin x,\ u_t(x,0)=\sin(3x) & 0<x<\pi \end{array}$$

Exercise 7.

$$\begin{array}{ll} u_{tt} = u_{xx} + \sin(2x)\cos t & 0<x<\pi,\ t>0 \\ u(0,t)=0,\ u(\pi,t)=0 & t>0 \\ u(x,0)=0,\ u_t(x,0)=\sin x & 0<x<\pi \end{array}$$

Exercise 8. The function $f(x, y)$ is doubly periodic with period 2π in x and in y. It is given on $[-\pi,\ \pi]^2$ by $f(x, y) = xy^2$. Find the double Fourier series of f.

Exercise 9. Same question as in problem 8 for the function given on $[-\pi,\ \pi]^2$ by $f(x, y) = x^2y^2$.

Exercise 10. Let $f(x, y) = 1$ on the square $[0,\ 1]^2$. Find

1. The Fourier cosine-cosine series of f.
2. The Fourier cosine-sine series of f.
3. The Fourier sine-sine series of f.
4. The Fourier sine-cosine series of f.

Exercise 11. Same questions as in problem 10 for the function $f(x, y) = xy$ on the square $[0,\ \pi]^2$.

Exercise 12. Find the Fourier sine-sine series of the function $f(x, y)$ given on the square $[0,\ \pi]^2$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x > y \end{cases}$$

In the remaining exercises use multiple Fourier series to solve the BVP (double series except in the last exercise where you can use triple Fourier series).

Exercise 13.

$$\begin{array}{ll} u_t = 4(u_{xx} + u_{yy}), & 0 < x < 2,\ 0 < y < 1,\ t > 0 \\ u_x(0, y, t) = u_x(2, y, t) = 0, & 0 < y < 1,\ t > 0 \\ u(x, 0, t) = u(x, 1, t) = 0, & 0 < x < 2,\ t > 0 \\ u(x, y, 0) = 100 & 0 < x < 2,\ 0 < y < 1\ . \end{array}$$

Exercise 14.

$$\begin{array}{ll} u_{tt} = u_{xx} + u_{yy}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\ u(0, y, t) = u(\pi, y, t) = 0, & 0 < y < \pi,\ t > 0 \\ u(x, 0, t) = u(x, \pi, t) = 0, & 0 < x < \pi,\ t > 0 \\ u(x, y, 0) = 0.05x(\pi - x)y(\pi - y) & 0 < x < \pi,\ 0 < y < \pi \\ u_t(x, y, 0) = 0 & 0 < x < \pi,\ 0 < y < \pi\ . \end{array}$$

Exercise 15.

$$\begin{array}{ll} u_{tt} = u_{xx} + u_{yy}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\ u(0, y, t) = u(\pi, y, t) = 0, & 0 < y < \pi,\ t > 0 \\ u(x, 0, t) = u(x, \pi, t) = 0, & 0 < x < \pi,\ t > 0 \\ u(x, y, 0) = 0 & 0 < x < \pi,\ 0 < y < \pi \\ u_t(x, y, 0) = f(x, y) & 0 < x < \pi,\ 0 < y < \pi\ . \end{array}$$

where

$$f(x, y) = \begin{cases} 1 & \text{if } \ \pi/4 < x < 3\pi/4,\ \pi/4 < y < 3\pi/4 \\ 0 & \text{elsewhere} \end{cases}$$

Exercise 16.

$$\begin{array}{ll} u_{xx} + u_{yy} = 2u + 1, & 0 < x < \pi, \quad 0 < y < \pi, \\ u(0, y) = u(\pi, y) = 0, & 0 < y < \pi, \\ u(x, 0) = u(x, \pi) = 0, & 0 < x < \pi\ . \end{array}$$

Exercise 17.

$$\begin{aligned}
&u_{xx}+u_{yy}=xy, && 0<x<\pi,\quad 0<y<\pi,\\
&u(0,y)=u(\pi,y)=0, && 0<y<\pi,\\
&u(x,0)=u(x,\pi)=0, && 0<x<\pi\ .
\end{aligned}$$

Exercise 18. (Dirichlet problem in a cube)

$$\begin{aligned}
&u_{xx}+u_{yy}+u_{zz}=0, && 0<x<\pi,\quad 0<y<\pi,\quad 0<z<\pi,\\
&u(0,y,z)=u(\pi,y,z)=0, && 0<y<\pi,\quad 0<z<\pi\\
&u(x,0,z)=-\sin(2x)\sin(5z), && 0<x<\pi\quad 0<z<\pi,\\
&u(x,\pi,z)=\sin(3x)\sin(z), && 0<x<\pi\quad 0<z<\pi,\\
&u(x,y,0)=\sin x\sin(2y),\quad u(x,y,\pi)=0, && 0<x<\pi,\quad 0<y<\pi\ .
\end{aligned}$$