FOURIER-BESSEL SERIES AND BOUNDARY VALUE PROBLEMS IN CYLINDRICAL COORDINATES

The parametric Bessel's equation appears in connection with the Laplace operator in polar coordinates. The method of separation of variables for problem with cylindrical geometry leads a singular Sturm-Liouville with the parametric Bessel's equation which in turn allows solutions to be represented as series involving Bessel functions.

1. THE PARAMETRIC BESSEL EQUATION

The parametric Bessel's equation of order α (with $\alpha \geq 0$) is the the second order ODE

$$(1) \qquad x^2 y'' + xy' + (\lambda x^2 - \alpha^2)y = 0 \,,$$

with $\lambda > 0$ a parameter, or equivalently,

$$(2) \qquad x^2 y'' + xy' + (\nu^2 x^2 - \alpha^2)y = 0 \,,$$

where $\nu = \sqrt{\lambda}$. We have the following lemma.

Lemma 1. *The general solution of equation* (1) *in $x > 0$ is*

$$y(x) = AJ_\alpha(\nu x) + BY_\alpha(\nu x) \,,$$

where J_α and Y_α are, respectively, the Bessel functions of the first and second kind of order α and A, B are constants. Moreover, $\lim_{x\to 0^+} y(x)$ is finite if and only if $B = 0$ and so

$$y(x) = AJ_\alpha(\nu x) \,.$$

Proof. We transform equation (2) into the standard Bessel equation of order α by using the substitution $t = \nu x$. Indeed, we have

$$\frac{dy}{dx} = \nu \frac{dy}{dt} \,, \quad \text{and} \quad \frac{d^2 y}{dx^2} = \nu^2 \frac{d^2 y}{dt^2} \,.$$

In terms of the variable t, equation (1) becomes

$$x^2 \nu^2 \frac{d^2 y}{dt^2} + x\nu \frac{dy}{dt} + (x^2 \nu^2 - \alpha^2)y = 0$$
$$t^2 \frac{d^2 y}{dt^2} + t\frac{dy}{dt} + (t^2 - \alpha^2)y = 0$$

which is the Bessel equation of order α in the variable t. The general solution is therefore, $y(t) = AJ_\alpha(t) + BY_\alpha(t)$. This is precisely,

$$y(x) = AJ_\alpha(\nu x) + BY_\alpha(\nu x) \,.$$

Note that $J_\alpha(0) = 0$ if $\alpha > 0$ and $J_0(0) = 1$, while the second solution Y_α satisfies $\lim_{x \to 0^+} Y_\alpha(x) = -\infty$. Hence, if the solution $y(x)$ is bounded in the interval $(0, \epsilon)$ (with $\epsilon > 0$), then necessarily $B = 0$.

We can rewrite equation (1) in a self-adjoint form by dividing by x and noticing that $(xy')' = xy'' + y'$ to obtain

$$(xy')' + \left(\lambda x - \frac{\alpha^2}{x}\right) y = 0 .$$

We have here (with the notation of Note 9) that $p(x) = x$, $r(x) = x$, and $q(x) = \alpha^2/x$. The point $x = 0$ is a singular point. Any Sturm-Liouville problem associated with this equation on the interval $[0, R]$ is a singular SL. As mentioned in Note 9, we can use use only the boundary condition at the endpoint $x = R$.

2. A Singular Sturm-Liouville Problem

We associate to the parametric Bessel equation the following SL-problem

$$(3) \qquad \begin{cases} (xy')' + \left(\lambda x - \dfrac{\alpha^2}{x}\right) y = 0 \\ b_1 y(R) + b_2 y'(R) = 0 \end{cases}$$

It is understood in this problem that $y(x)$ is a bounded solution, or equivalently that $\lim_{x \to 0^+} y(x)$ exists and is finite. In the space $C_p^0[0, R]$, of piecewise continuous function on $[0, R]$, we consider the following inner product

$$< f, g >_x = \int_0^R f(x)g(x)x\,dx .$$

The proof given in Theorem 2 of Note 9, can be carried out almost verbatim to establish the orthogonality of the eigenfunctions of problem (3).

There is an exceptional case which occurs when $b_1 = -b_2\alpha/R$ (with $b_2 \neq 0$). In this case $\lambda = 0$ is an eigenvalue of Problem (3) with an eigenfunction x^α. To simplify the discussion we will assume that $b_1 \neq -b_2\alpha/R$.

By Lemma 1, we know that the bounded solutions of the ODE are $y(x) = J_\alpha(\nu x)$ (with $\nu = \sqrt{\lambda}$). Note that $y'(x) = \nu J_\alpha'(\nu x)$ In order for such a solution to satisfy the boundary condition, the parameter ν must satisfy

$$(4) \qquad b_1 J_\alpha(\nu R) + b_2 \nu J_\alpha'(\nu R) = 0 .$$

This means that $\lambda = \nu^2$ is an eigenvalue of Problem (3) with eigenfunction $J_\alpha(\nu x)$. It can also be proved that these solutions form a complete set (basis) in the space of piecewise continuous functions. This imply that any function $f \in C_p^0[0, R]$ has a representation as a series in these eigenfunctions. Denote by

$$0 \leq \nu_1 < \nu_2 < \nu_3 < \cdots < \nu_j < \cdots ,$$

be the increasing sequence of solutions of equation (4). The series expansion is then

$$f(x) \sim \sum_{j=1}^{\infty} C_j J_\alpha(\nu_j x) ,$$

where the coefficients are given by,

$$C_j = \frac{< f, J_\alpha(\nu_j x) >_x}{||J_\alpha(\nu_j x)||_x^2} = \left(\int_0^R f(x)J_\alpha(\nu_j x)x\,dx\right) \bigg/ \left(\int_0^R J_\alpha(\nu_j x)^2 x\,dx\right)$$

It is of importance to find the square norms $||J_\alpha(\nu_j x)||_x^2$. We have the lemma.

Lemma 2. *Let ν_j be a positive solution of* (4), *then we have the following:*

- *If $b_2 = 0$, then*

$$||J_\alpha(\nu_j x)||_x^2 = \int_0^R J_\alpha(\nu_j x)^2 x dx = \frac{R^2}{2} J_{\alpha+1}(\nu_j R)^2$$

- *If $b_2 \neq 0$, let $h = Rb1/b_2$, then we have*

$$||J_\alpha(\nu_j x)||_x^2 = \int_0^R J_\alpha(\nu_j x)^2 x dx = \frac{R^2\nu_j^2 - \alpha^2 + h^2}{2\nu_j^2} J_\alpha(\nu_j R)^2$$

Proof. Since $J_\alpha(\nu_j x)$ satisfies Problem (3) with $\lambda = \nu_j^2$, then

$$x(x J_\alpha(\nu_j x)')' = (\alpha^2 - \nu_j^2 x^2) J_\alpha(\nu_j x) .$$

We multiply by $2 J_\alpha(\nu_j x)'$ to get

$$2x J_\alpha(\nu_j x)' \left[x J_\alpha(\nu_j x)' \right]' = (\alpha^2 - \nu_j^2 x^2) 2 J_\alpha(\nu_j x) J_\alpha(\nu_j x)' .$$

By using $2ff' = (f^2)'$, we obtain

$$\left[(x J_\alpha(\nu_j x)')^2 \right]' = (\alpha^2 - \nu_j^2 x^2) \left(J_\alpha(\nu_j x)^2 \right)' .$$

Now we integrate from 0 to R. The left side is

$$\int_0^R \left[(x J_\alpha(\nu_j x)')^2 \right]' dx = \left[(x J_\alpha(\nu_j x)')^2 \right]_{x=0}^{x=R} = (R J_\alpha(\nu_j R)')^2 .$$

For the right side we use integration by parts,

$$\int_0^R (\alpha^2 - \nu_j^2 x^2) \left(J_\alpha(\nu_j x)^2 \right)' dx = \left[(\alpha^2 - \nu_j^2 x^2) J_\alpha(\nu_j x)^2 \right]_0^R + 2\nu_j^2 \int_0^R J_\alpha(\nu_j x)^2 x dx$$
$$= 2\nu_j^2 ||J_\alpha(\nu_j x)||_x^2 + (\alpha^2 - \nu_j^2 R^2) J_\alpha(\nu_j R)^2$$

After equating the two sides, we get

$$(5) \qquad 2\nu_j^2 ||J_\alpha(\nu_j x)||_x^2 = R^2 (J_\alpha(\nu_j R)')^2 + (\nu_j^2 R^2 - \alpha^2) J_\alpha(\nu_j R)^2$$

We distinguish the two cases.

If $b_2 = 0$, then the condition is $J_\alpha(\nu_j R) = 0$, and relation (5) becomes

$$||J_\alpha(\nu_j x)||_x^2 = \frac{R^2 (J_\alpha(\nu_j R)')^2}{2\nu_j^2} .$$

We simplify this expression by using Property (4) of Bessel junctions $((x^{-\alpha} J_\alpha(x))' = -x^{-\alpha} J_{\alpha+1}(x))$ in the form

$$J_\alpha(x)' - \frac{\alpha}{x} J_\alpha(x) = -J_{\alpha+1}(x) .$$

For $x = \nu_j R$, we have $J_\alpha(\nu_j R) = 0$ and it follows that

$$||J_\alpha(\nu_j x)||_x^2 = \frac{R^2 J_{\alpha+1}(\nu_j R)^2}{2\nu_j^2} .$$

If $b_2 \neq 0$, then the boundary condition can be written as

$$\nu_j J_\alpha'(\nu_j R) = -\frac{b_1}{b_2} J_\alpha(\nu_j R) = -\frac{h}{R} J_\alpha(\nu_j R) .$$

Relation (5) becomes

$$2\nu_j^2 ||J_\alpha(\nu_j x)||_x^2 = h^2 J_\alpha(\nu_j R)^2 + (\nu_j^2 R^2 - \alpha^2) J_\alpha(\nu_j R)^2 = (R^2\nu_j^2 + h^2 - \alpha^2) J_\alpha(\nu_j R)^2 .$$

From this we deduce that

$$||J_\alpha(\nu_j x)||_x^2 = \frac{R^2\nu_j^2 + h^2 - \alpha^2}{2\nu_j^2} J_\alpha(\nu_j R)^2 .$$

3. Bessel Series Expansions

We summarize the above discussion in the following theorems.

Theorem 1. *Consider the SL-problem*

$$x^2 y'' + xy' + (\lambda x^2 - \alpha^2)y = 0, \qquad 0 < x < R ,$$
$$y(R) = 0$$

with bounded solutions $y(x)$. The eigenvalues values consist of the sequence $\lambda_j = \nu_j^2$, with $j \in \mathbb{Z}^+$ and where $\nu_j R$ is the j-th positive zero of $J_\alpha(x)$. That is,

$$\nu_1 < \nu_2 < \nu_3 < \cdots < \nu_j < \cdots$$

satisfy $J_\alpha(\nu_j R) = 0$. The corresponding j-th eigenfunction is

$$y_j(x) = J_\alpha(\nu_j x).$$

Furthermore, the family of eigenfunctions $J_\alpha(\nu_j x)$ is complete in $C_p^0[0, \ R]$. The square norm of the j-th eigenfunction is

$$||J_\alpha(\nu_j x)||_x^2 = \int_0^R J_\alpha(\nu_j x)^2 x dx = \frac{R^2}{2} J_{\alpha+1}(\nu_j R)^2$$

Theorem 2. *Consider the SL-problem*

$$x^2 y'' + xy' + (\lambda x^2 - \alpha^2)y = 0, \qquad 0 < x < R ,$$
$$b_1 y(R) + b_2 y'(R) = 0, \qquad (b_2 \neq 0 \quad b_1 \geq (-\alpha b_2/R))$$

with bounded solutions $y(x)$. The eigenvalues consist of the sequence $\lambda_j = \nu_j^2$, with $j \in \mathbb{Z}^+$, such that,

$$\nu_1 < \nu_2 < \nu_3 < \cdots < \nu_j < \cdots$$

satisfy the equation

$$h J_\alpha(\nu_j R) + \nu_j R J'_\alpha(\nu_j R) = 0 , \qquad h = \frac{R b_1}{b_2}$$

The corresponding j-th eigenfunction is

$$y_j(x) = J_\alpha(\nu_j x).$$

Furthermore, the family of eigenfunctions $J_\alpha(\nu_j x)$ is complete in $C_p^1[0, \ R]$. The square norm of the j-th eigenfunction is

$$||J_\alpha(\nu_j x)||_x^2 = \int_0^R J_\alpha(\nu_j x)^2 x dx = \frac{R^2\nu_j^2 - \alpha^2 + h^2}{2\nu_j^2} J_\alpha(\nu_j R)^2$$

Consequence of Theorem 1 and 2. *If $f \in C_p^0[0, \ R]$, then f has the expansion,*

$$f_{av}(x) = \frac{f(x^+) + f(x^-)}{2} = \sum_{j=1}^\infty C_j J_\alpha(\nu_j x) ,$$

with

$$C_j = \frac{< f(x), J_\alpha(\nu_j x) >_x}{||J_\alpha(\nu_j x)||_x^2}$$

and where the functions $J_\alpha(\nu_j x)$ are as in Theorem 1 or as in Theorem 2.

Remark 1. If we set $z_j = \nu_j R$, then the above series takes the form

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{j=1}^{\infty} C_j J_\alpha \left(z_j \frac{x}{R} \right)$$

where in the case of Theorem 1, the z_j's are the positive roots of J_α, i.e.,

$$J_\alpha(z_j) = 0, \qquad j = 1,\ 2,\ 3,\ \cdots,$$

in the case of Theorem 2, the z_j's are the positive roots of the equation

$$h J_\alpha(z_j) + z_j J_\alpha'(z_j) = 0, \qquad j = 1,\ 2,\ 3,\ \cdots.$$

Remark 2. Since $J_\alpha(0) = 0$ for $\alpha > 0$ and $J_0(0) = 1$, then $\lambda = 0$ is an eigenvalue of the SL-problem (3) only in the case of boundary condition $y'(R) = 0$ and the eigenfunction is $J_0(z_j x/R)$, where the z_j's form the sequence of positive zeros of J_0'.

We illustrate this theorems with some examples.

Example 1. Expand the function $f(x) = 1$ on $[0,\ 3]$ into a J_0-Bessel series with condition $J_0(z_j) = 0$. This is the case of Theorem 1. We have

$$1 = \sum_{j=1}^{\infty} C_j J_0 \left(\frac{z_j x}{3} \right) \qquad 0 < x < 3,$$

Where

$$C_j = \frac{1}{||J_0(z_j x/3)||_x^2} \int_0^3 J_0(z_j x/3) x dx .$$

The square norms are $||J_0(z_j x/3)||_x^2 = 9 J_1(z_j)^2/2$ and to find the integral we use Property 9 ($\int t^\alpha J_{\alpha-1}(t) dt = t^\alpha J_\alpha(t) + C$ and a substitution).

$$\int_0^3 J_0(z_j x/3) x dx = \frac{9}{z_j^2} \int_0^{z_j} t J_0(t) dt = \frac{9}{z_j^2} \left(z_j J_1(z_j) - 0 J_1(0) \right) = \frac{9 J_1(z_j)}{z_j} .$$

Hence,

$$C_j = \frac{2}{z_j J_1(z_j)}$$

and the series expansion is

$$1 = 2 \sum_{j=1}^{\infty} \frac{1}{z_j J_1(z_j)} J_0 \left(\frac{z_j x}{3} \right) \qquad 0 < x < 3 .$$

Now we give approximations of the first five terms of the series

j	1	2	3	4	5
z_j	2.405	5.520	8.654	11.792	14.931
$J_1(z_j)$	0.519	-0.340	0.272	-0.233	0.207
$2/z_j J_1(z_j)$	0.801	-0.532	0.426	-0.365	0.324

The beginning of the series expansion is

$$1 \approx 0.801 J_0(0.80x) - 0.532 J_0(1.84x) + 0.426 J_0(2.88x)$$
$$-0.365 J_0(3.93x) + 0.324 J_0(4.98x) + \cdots$$

Example 2. An analogous calculation gives the expansion of $f(x) = 1$ over $[0, \ R]$ with respect J_α with end point condition $J_\alpha(z_j) = 0$ as

$$1 = \sum_{j=1}^{\infty} C_j J_\alpha\left(\frac{z_j x}{R}\right) \qquad 0 < x < R,$$

Where

$$C_j = \frac{2}{R^2 J_{\alpha+1}(z_j)^2} \int_0^R J_\alpha(z_j x/R) x dx = \frac{2}{z_j^2 J_{\alpha+1}(z_j)^2} \int_0^{z_j} t J_\alpha(t) dt \ .$$

Example 3. Expand $f(x) = 1$ on $[0, \ 2]$ in J_α-Bessel series with endpoint condition $J_\alpha'(z_j) = 0$. This is the case of Theorem 2 with $R = 2$ and $h = 0$. The square norms are

$$||J_\alpha(z_j x/2)||_x^2 = \frac{2(z_j^2 - \alpha^2)}{z_j^2} J_\alpha(z_j)^2$$

and

$$1 = \sum_{j=1}^{\infty} C_j J_\alpha\left(\frac{z_j x}{2}\right) ,$$

with

$$C_j = \frac{z_j^2}{2(z_j^2 - \alpha^2) J_\alpha(z_j)^2} \int_0^2 x J_\alpha(z_j x/2) dx = \frac{2}{(z_j^2 - \alpha^2) J_\alpha(z_j)^2} \int_0^{z_j} t J_\alpha(t) dt$$

When $\alpha = 0$, we get a more compact form of C_j

$$C_j = \frac{2}{z_j^2 J_0(z_j)^2} \int_0^{z_j} t J_0(t) dt = \frac{2}{z_j^2 J_0(z_j)^2} [t J_1(t)]_0^{z_j} = \frac{2 J_1(z_j)}{z_j J_0(z_j)^2}$$

The expansion of $f(x) = 1$ over $[0, \ 2]$ in a Bessel series with endpoint $J_0'(z_j) = 0$ is

$$1 = 2 \sum_{j=1}^{\infty} \frac{J_1(z_j)}{z_j J_0(z_j)^2} J_0\left(\frac{z_j x}{2}\right) \ .$$

Example 4. Find the J_0-Bessel series of $f(x) = 1 - x^2$ over $[0, \ 1]$ with endpoint $J_0(z_j) = 0$.

We have for $0 < x < 1$,

$$(1 - x^2) = \sum_{j=1}^{\infty} C_j J_0(z_j x), \qquad C_j = \frac{< 1 - x^2, J_0(z_j x) >_x}{||J_0(z_j x)||_x^2} \ .$$

We know that $||J_0(z_j x)||_x^2 = \dfrac{J_1(z_j)^2}{2}$ (see previous calculations). To compute $< 1 - x^2, J_0(z_j x) >_x$, we use the substitution $t = z_j x$ in the integral and the

properties $(t^\alpha J_\alpha(t))' = t^\alpha J_{\alpha-1}(t)$ and $(t^{-\alpha} J_\alpha(t))' = -t^{-\alpha} J_{\alpha+1}(t)$:

$$
\begin{aligned}
<1 - x^2, J_0(z_j x) >_x &= \frac{1}{z_j^4} \int_0^{z_j} (z_j^2 - t^2) t J_0(t) dt = \frac{1}{z_j^4} \int_0^{z_j} (z_j^2 - t^2)(t J_1(t))' dt \\
&= \frac{\left[(z_j^2 - t^2) t J_1(t)\right]_0^{z_j}}{z_j^4} + \frac{2}{z_j^4} \int_0^{z_j} t^2 J_1(t) dt \\
&= \frac{2}{z_j^4} \int_0^{z_j} t^2 J_1(t) dt = \frac{2}{z_j^4} \int_0^{z_j} t^2 (-J_0(t))' dt \\
&= \frac{2 \left[-t^2 J_0(t)\right]_0^{z_j}}{z_j^4} + \frac{4}{z_j^4} \int_0^{z_j} t J_0(t) dt = \frac{4}{z_j^4} \int_0^{z_j} t J_0(t) dt \\
&= \frac{4}{z_j^4} \left[t J_1(t)\right]_0^{z_j} = \frac{4 J_1(z_j)}{z_j^3}
\end{aligned}
$$

Hence the coefficient C_j is

$$
C_j = \frac{4 J_1(z_j)/z_j^3}{J_1(z_j)^2/2} = \frac{8}{z_j^3 J_1(z_j)} \ .
$$

The series representation of $(1 - x^2)$ is

$$
(1 - x^2) = 9 \sum_{j=1}^{\infty} \frac{J_0(z_j x)}{z_j^3 J_1(z_j)}, \qquad x \in (0,\ 1)\ .
$$

Example 5. The expansion of x^m over $[0,\ 1]$ in J_m-Bessel series with endpoint condition $J_m(z_j) = 0$ can be obtained by using similar arguments as before.

$$
x^m = \sum_{j=1}^{\infty} C_j J_m(z_j x), \qquad C_j = \frac{< x^m, J_m(z_j x) >_x}{||J_m(z_j x)||_x^2}
$$

A calculation gives

$$
C_j = \frac{2}{z_j J_{m+1}(z_j)}
$$

and the series is

$$
x^m = 2 \sum_{j=1}^{\infty} \frac{J_m(z_j x)}{z_j J_{m+1}(z_j)}\ , \qquad x \in (0,\ 1)\ .
$$

4. Vibrations of a Circular Membrane

The following wave propagation problem models the small vertical vibrations of a circular membrane of radius L whose boundary is held fixed. The initial position and velocity are given by the functions f and g.

$$
\begin{aligned}
u_{tt}(x, y, t) &= c^2 \Delta u(x, y, t) & x^2 + y^2 &< L^2\,,\ t > 0\,, \\
u(x, y, t) &= 0 & x^2 + y^2 &= L^2\,,\ t > 0\,, \\
u(x, y, 0) &= f(x, y) & x^2 + y^2 &< L^2\,, \\
u_t(x, y, 0) &= g(x, y) & x^2 + y^2 &< L^2\,.
\end{aligned}
$$

where

$$
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
$$

is the 2-dimensional Laplace operator. For the purpose of using the method of separation of variables, this problem is better suited for polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Recall the expression of the Laplace operator in polar coordinates is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \; .$$

The expression of the BVP becomes

(6)
$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2}\right) & r < L, \; 0 \le \theta \le 2\pi, \; t > 0, \\ u(L,\theta,t) = 0 & 0 \le \theta \le 2\pi, \; t > 0, \\ u(r,\theta,0) = f(r,\theta) & r < L, \; 0 \le \theta \le 2\pi, \\ u_t(r,\theta,0) = g(r,\theta) & r < L, \; 0 \le \theta \le 2\pi, . \end{cases}$$

It is understood that the sought solution $u(r,\theta,t)$ and its derivatives need to be periodic in θ: i.e., $u(r,\theta,t) = u(r,\theta+2\pi,t)$ and so on. Since in this form the PDE has a singularity at $r = 0$, we impose that the solution u be continuous at the origin.

Before we consider the general case, we are going to solve problem (6) in the particular case when f and g are independent on θ (invariant under rotations).

4.1. **Rotationally invariant case.** Since f and g depend only on r, we seek solution u that is also independent on θ. That is, at each time t the displacement , $u = u(r,t)$ depends only on the radius r and not on the angle θ. In this case problem (6) becomes a problem in 2 variables:

(7)
$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \dfrac{u_r}{r}\right) & r < L, \; t > 0, \\ u(L,t) = 0 & t > 0, \\ u(r,0) = f(r) & r < L, \\ u_t(r,0) = g(r) & r < L. \end{cases}$$

We seek nontrivial solutions with separated variables $u = R(r)T(t)$ of the homogeneous part of the problem:

$$u_{tt} = c^2\left(u_{rr} + \frac{u_r}{r}\right), \qquad u(L,t) = 0 \; .$$

The separation of variables of the PDE leads to

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} = \frac{T''(t)}{c^2 T(t)} = -\lambda \qquad \text{(constant)} \; .$$

This together with the boundary condition give the ODE problems for $R(r)$ and $T(t)$

$$\begin{cases} r^2 R'' + rR' + \lambda r^2 R = 0, & 0 < r < R, \\ R(L) = 0 \end{cases}$$

and

$$T'' + c^2 \lambda T = 0 \; .$$

The R-problem is the eigenvalue problem. Its ODE is the parametric Bessel equation of order 0. Since we are seeking bounded solutions, then the eigenvalues and eigenfunctions are

$$\lambda_j = \frac{z_j^2}{L^2}, \quad R_j(r) = J_0\left(\frac{z_j r}{L}\right), \quad j = 1, \, 2, \, 3, \, \cdots$$

where z_j is the j-th positive root of $J_0(x) = 0$.

For each $j \in \mathbb{Z}^+$, the T-equations becomes $T'' + (cz_j/L)^2 T = 0$ with independent solutions $\cos(cz_j t/L)$ and $\sin(cz_j t/L)$.

For each $j \in \mathbb{Z}^+$, the homogeneous part has two independent solutions with separated variables

$$u_j^1(r,t) = \cos\left(\frac{cz_j t}{L}\right) J_0\left(\frac{z_j r}{L}\right) \quad \text{and} \quad u_j^2(r,t) = \sin\left(\frac{cz_j t}{L}\right) J_0\left(\frac{z_j r}{L}\right).$$

The general series solution of the homogeneous part is the linear combination of all the u_j^1's and u_j^2's:

$$u(r,t) = \sum_{j=1}^{\infty} \left[A_j \cos\left(\frac{cz_j t}{L}\right) + B_j \sin\left(\frac{cz_j t}{L}\right) \right] J_0\left(\frac{z_j r}{L}\right)$$

Now we use the nonhomogeneous conditions to find the constants A_j and B_j so that the series solution satisfies the whole problem. First we compute u_t:

$$u_t(r,t) = \sum_{j=1}^{\infty} \frac{cz_j}{L} \left[-A_j \sin\left(\frac{cz_j t}{L}\right) + B_j \cos\left(\frac{cz_j t}{L}\right) \right] J_0\left(\frac{z_j r}{L}\right).$$

We have

$$u(r,0) = f(r) = \sum_{j=1}^{\infty} A_j J_0\left(\frac{z_j r}{L}\right),$$

$$u_t(r,0) = g(r) = \sum_{j=1}^{\infty} \frac{cz_j}{L} B_j J_0\left(\frac{z_j r}{L}\right).$$

These are, respectively, the J_0-Bessel series of the functions f and g over $[0,\ L]$ with endpoint condition $J_0(z_j) = 0$. We have then,

$$A_j = \frac{< f(r), J_0(z_j r/L) >_r}{||J_0(z_j r/L)||_r^2} = \frac{2}{L^2 J_1(z_j)^2} \int_0^L r f(r) J_0(z_j r/L) dr,$$

$$B_j = \frac{L}{cz_j} \frac{< g(r), J_0(z_j r/L) >_r}{||J_0(z_j r/L)||_r^2} = \frac{2}{cL z_j J_1(z_j)^2} \int_0^L r g(r) J_0(z_j r/L) dr.$$

Example. (The struck drum head). Suppose that the above circular membrane, with $L = 10$ which was originally at equilibrium is struck at time $t = 0$ at its center in such a way that each point located at distance < 1 from the center is given a velocity -1. The BVP in this case is (we take $c = 1$)

$$\begin{cases} u_{tt} = \left(u_{rr} + \dfrac{u_r}{r} \right) & r < 10,\ t > 0, \\ u(10,t) = 0 & t > 0, \\ u(r,0) = 0 \quad u_t(r,0) = g(r) & r < L. \end{cases}$$

where

$$g(r) = \begin{cases} -1 & \text{if } 0 < r < 1, \\ 0 & \text{if } 1 < r < 10. \end{cases}$$

In this case the coefficients are $A_j = 0$ and

$$B_j = \frac{1}{5 z_j J_1(z_j)^2} \int_0^{10} r g(r) J_0(z_j r/10) dr.$$

For the last integral, we have

$$\int_0^{10} rg(r)J_0(z_jr/10)dr \quad = -\int_0^1 rJ_0(z_jr/10)dr = -\frac{100}{z_j^2}\int_0^{z_j/10} tJ_0(t)dt$$
$$= -\frac{100}{z_j^2}\left[tJ_1(t)\right]_0^{z_j/10} = -\frac{10}{z_j}J_1(z_j/10)$$

We have therefore

$$B_j = -\frac{2J_1(z_j/10)}{z_j^2 J_1(z_j)^2}$$

and

$$u(r,t) = -2\sum_{j=1}^{\infty}\frac{J_1(z_j/10)}{z_j^2 J_1(z_j)^2}\sin\left(\frac{z_jt}{10}\right)J_0\left(\frac{z_jr}{10}\right) .$$

The following table list the approximate values

j	1	2	3	4	5
z_j	2.405	5.520	8.654	11.792	14.931
$J_1(z_j)$	0.519	-0.340	0.272	-0.233	0.207
$J_1(z_j/10)$	0.119	0.266	0.393	0.493	0.557
B_j	-0.153	-0.151	-0.143	-0.131	-0.117

We have then

$$u(r,t) \approx -0.153\sin(0.24t)J_0(0.24r) - 0.151\sin(0.55t)J_0(0.55r)$$
$$-0.143\sin(0.86t)J_0(0.86r) - 0.131\sin(1.18t)J_0(1.18r)$$
$$-0.117\sin(14.9t)J_0(14.9r) - \cdots$$

4.2. **General case.** Consider again BVP (6). This time we assume that the initial position f and initial velocity g depend effectively on θ and r and also $u = u(r,\theta,t)$. We emphasize the 2π-periodicity of u and its derivative u_θ, with respect to θ, by adding it to the problem. The problem is therefore

$$(8)\quad\begin{cases} u_{tt} = c^2\left(u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2}\right) & r < L,\ 0 \le \theta \le 2\pi,\ t > 0, \\ u(L,\theta,t) = 0 & 0 \le \theta \le 2\pi,\ t > 0, \\ u(r,-\pi,t) = u(r,\pi,t) & 0 < r < L,\ t > 0, \\ u_\theta(r,-\pi,t) = u_\theta(r,\pi,t) & 0 < r < L,\ t > 0, \\ u(r,\theta,0) = f(r,\theta) & r < L,\ 0 \le \theta \le 2\pi, \\ u_t(r,\theta,0) = g(r,\theta) & r < L,\ 0 \le \theta \le 2\pi. \end{cases}$$

We apply the method of separation of variables to the homogeneous part. Let

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

be a nontrivial solution of the homogeneous part. For such a function the wave equation can be written as

$$\frac{T''(t)}{c^2 T(t)} = \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = -\lambda \quad \text{(constant)} .$$

Thus $T''(t) + c^2\lambda T(t) = 0$ and

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} - \lambda r^2\frac{\Theta''(\theta)}{\Theta(\theta)} = 0 .$$

The last equation can be separated:

$$r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \lambda r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \alpha^2 \qquad \text{(constant)} \,.$$

The ODEs for R and Θ are

$$r^2 R'' + rR' + (\lambda r^2 - \alpha^2)R = 0 \quad \text{and} \quad \Theta'' + \alpha^2\Theta = 0 \,.$$

The homogeneous boundary conditions imply that $R(L) = 0$, $\Theta(\pi) = \Theta(-\pi)$ and $\Theta'(\pi) = \Theta'(-\pi)$. The three ODE problems are

$$\begin{cases} \Theta'' + \alpha^2\Theta = 0 \\ \Theta(\pi) = \Theta(-\pi) \\ \Theta'(\pi) = \Theta'(-\pi) \end{cases} \qquad \begin{cases} r^2 R'' + rR' + (\lambda r^2 - \alpha^2)R = 0 \\ R(L) = 0 \end{cases} \qquad T''(t) + c^2\lambda T(t) = 0$$

The Θ-problem is the regular periodic SL-problem and the R-problem is the singular Bessel problem of order α with endpoint condition $R(L) = 0$.

The eigenvalues and eigenfunctions of the Θ-problem are.

$$\alpha_0^2 = 0\,, \quad \Theta_0(\theta) = 1 \,;$$

for $m \in \mathbb{Z}^+$,

$$\alpha_m^2 = m^2\,, \quad \Theta_m^1(\theta) = \cos(m\theta)\,, \quad \Theta_m^2(\theta) = \sin(m\theta)\,.$$

We move to the R-problem. For each $\alpha = m$ (with $m \in \mathbb{Z}^+ \cup \{0\}$), the eigenvalues and eigenfunctions of the R-problem are

$$\lambda_{mj} = \frac{z_{mj}^2}{L^2}\,, \quad R_{mj}(r) = J_m\left(\frac{z_{mj}r}{L}\right)\,,$$

where $\{z_{mj}\}_j$ is the increasing sequence of zeros of J_m. That is,

$$0 < z_{m1} < z_{m2} < \cdots < z_{mj} < \cdots$$

satisfy $J_m(z_{mj}) = 0$.

Now that we have the eigenvalues and eigenfunctions of the Θ- and R-problems, we solve the T-problem. For each $m \in \mathbb{Z}^+ \cup \{0\}$ and each $j \in \mathbb{Z}^+$, the corresponding T-equation is

$$T'' + \left(\frac{cz_{mj}}{L}\right)^2 T = 0$$

with two independent solutions

$$T_{mj}^1 = \cos\left(\frac{cz_{mj}t}{L}\right) \quad \text{and} \quad T_{mj}^2 = \sin\left(\frac{cz_{mj}t}{L}\right) \,.$$

The collection of solutions with separated variables of the homogeneous part of Problem (8) consists of the following functions (where $m = 0,\ 1,\ 2,\ \cdots$ and $j = 1,\ 2,\ \cdots$)

$$u_{mj}^1(r,\theta,t) = J_m\left(\frac{z_{mj}r}{L}\right)\cos(m\theta)\cos\left(\frac{cz_{mj}t}{L}\right)\,,$$

$$u_{mj}^2(r,\theta,t) = J_m\left(\frac{z_{mj}r}{L}\right)\sin(m\theta)\cos\left(\frac{cz_{mj}t}{L}\right)\,,$$

$$u_{mj}^3(r,\theta,t) = J_m\left(\frac{z_{mj}r}{L}\right)\cos(m\theta)\sin\left(\frac{cz_{mj}t}{L}\right)\,,$$

$$u_{mj}^4(r,\theta,t) = J_m\left(\frac{z_{mj}r}{L}\right)\sin(m\theta)\sin\left(\frac{cz_{mj}t}{L}\right)\,.$$

These are the (m, j)-modes of vibrations of the membrane. Up to a rotation, the profile of each mode is given by the graph of the function

$$v_{mj}(r, \theta) = J_m \left(\frac{z_{mj} r}{L} \right) \cos(m\theta)$$

(each (m, j)-mode is just v_{mj} times a time-amplitude). For each (m, j), the membrane can be divided into regions where $v_{mj} > 0$ and regions where $v_{mj} < 0$. The curves where $v_{mj} = 0$ are called the *nodal lines* (see figure).
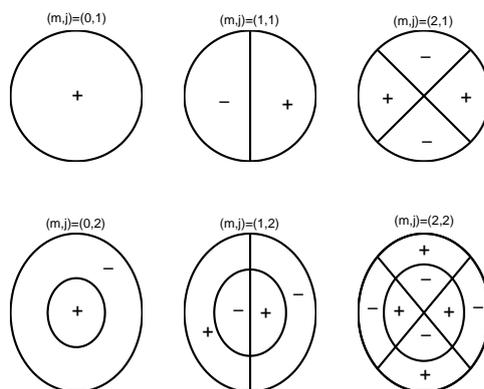
Some of the (m, j)-modes are graphed in the figure.
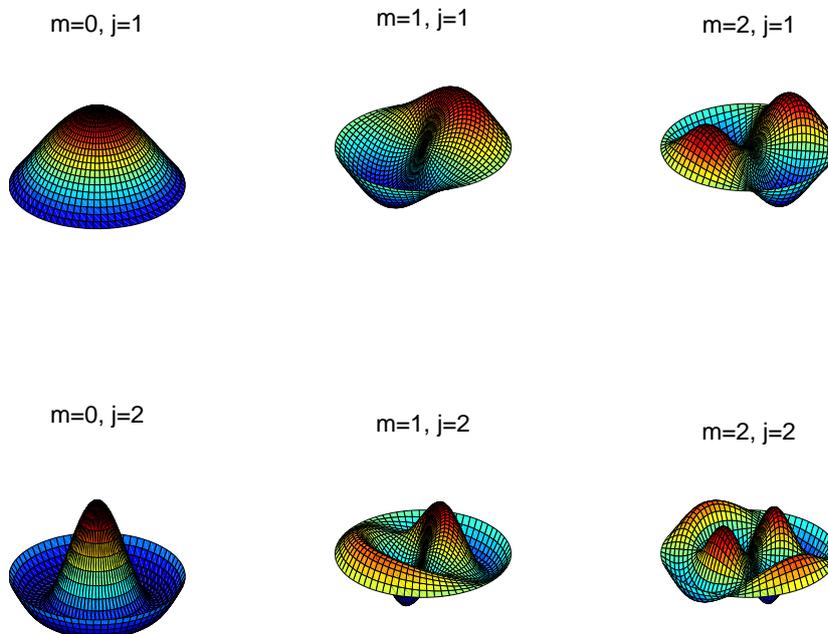
The general series solution of the homogeneous part is a linear combination of the solutions u_{mj}^k:

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} J_m \left(\frac{z_{mj}r}{L} \right) \cos \left(\frac{cz_{mj}t}{L} \right) (A_{mj} \cos(m\theta) + B_{mj} \sin(m\theta))$$

$$+ J_m \left(\frac{z_{mj}r}{L} \right) \sin \left(\frac{cz_{mj}t}{L} \right) (C_{mj} \cos(m\theta) + D_{mj} \sin(m\theta))$$

In order for such a solution to satisfy the nonhomogeneous condition, we need

$$u(r, \theta, 0) = f(r, \theta) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} J_m \left(\frac{z_{mj}r}{L} \right) [A_{mj} \cos(m\theta) + B_{mj} \sin(m\theta)]$$

and

$$u_t(r, \theta, 0) = g(r, \theta) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \frac{cz_{mj}t}{L} J_m \left(\frac{z_{mj}r}{L} \right) [C_{mj} \cos(m\theta) + D_{mj} \sin(m\theta)]$$

These are the (double) Fourier-Bessel series of f and g. To find the coefficients, we can proceed as follows. Expand $f(r, \theta)$ into its Fourier series (view r as a parameter),

$$f(r, \theta) = \frac{A_0(r)}{2} + \sum_{m=1}^{\infty} A_m(r) \cos(m\theta) + B_m(r) \sin(m\theta)$$

where

$$A_m(r) = \frac{1}{\pi} \int_0^{2\pi} f(r, \theta) \cos(m\theta) d\theta, \qquad m = 0, 1, 2, \cdots$$

$$B_m(r) = \frac{1}{\pi} \int_0^{2\pi} f(r, \theta) \sin(m\theta) d\theta, \qquad m = 1, 2, 3, \cdots$$

Now expand $A_m(r)$ and $B_m(r)$ into a J_m-Bessel series with end point condition $J_m(z) = 0$. We get,

$$A_m(r) = \sum_{j=1}^{\infty} A_{mj} J_m \left(\frac{z_{mj}r}{L} \right), \qquad A_{mj} = \frac{1}{||J_m||_r^2} \int_0^L r A_m(r) J_m \left(\frac{z_{mj}r}{L} \right) dr$$

and

$$B_m(r) = \sum_{j=1}^{\infty} B_{mj} J_m \left(\frac{z_{mj}r}{L} \right), \qquad B_{mj} = \frac{1}{||J_m||_r^2} \int_0^L r B_m(r) J_m \left(\frac{z_{mj}r}{L} \right) dr$$

After substituting the expressions of $A_m(r)$ and $B_m(r)$ in term of the integrals of f, we obtain the coefficients A_{mj} and B_{mj} of the series solution as:

$$A_{mj} = \frac{1}{\pi ||J_m(z_{mj}r/L)||_r^2} \int_0^{2\pi} \int_0^L r f(r, \theta) J_m \left(\frac{z_{mj}r}{L} \right) \cos(m\theta) dr d\theta,$$

$$B_{mj} = \frac{1}{\pi ||J_m(z_{mj}r/L)||_r^2} \int_0^{2\pi} \int_0^L r f(r, \theta) J_m \left(\frac{z_{mj}r}{L} \right) \sin(m\theta) dr d\theta.$$

The other coefficients C_{mj} and D_{mj} are obtained in a similar way by using the second initial condition.

Example Take $L = 1$, $c = 2$, $g(r, \theta) = 0$ and $f(r, \theta) = (1 - r^2) + \sin\theta$. In this situation $C_{mj} = D_{mj} = 0$ for $m \geq 0$ and $j \geq 1$. Furthermore,

$$A_{mj} = 0 \quad \text{for} \quad m \geq 1$$

and

$$B_{mj} = 0 \quad \text{for} \quad m \geq 2$$

Hence, the solution has the form

$$u(r, \theta, t) = \sum_{j \geq 1} A_{0j} J_0(z_{0j}r) \cos(2z_{0j}t) + \sum_{j \geq 1} B_{1j} J_1(z_{1j}r) \cos(2z_{1j}t) \sin \theta \,,$$

where

$$A_{0j} = \frac{1}{||J_0(z_{0j}r)||_r^2} \int_0^1 r(1 - r^2) J_0(z_{0j}r) dr$$

and

$$B_{1j} = \frac{1}{\pi ||J_1(z_{1j}r)||_r^2} \int_0^{2\pi} \int_0^1 r J_1(z_{1j}r) \sin^2 \theta \, dr d\theta = \frac{1}{||J_1(z_{1j}r)||_r^2} \int_0^1 r J_1(z_{1j}r) dr \,.$$

The square norms are

$$||J_0(z_{0j}r)||_r^2 = \frac{J_1(z_{0j})^2}{2} \quad \text{and} \quad ||J_1(z_{1j}r)||_r^2 = \frac{J_2(z_{1j})^2}{2} \,.$$

The integral involved in A_{0j} can be evaluated. First, notice that by using the property $(t^\alpha J_\alpha(t))' = t^\alpha J_{\alpha-1}(t)$ and integration by parts, we get

$$\int t^3 J_0(t) dt = t^2(t J_1(t)) - 2 \int t(t J_1(t)) dt = t^3 J_1(t) - 2t^2 J_2(t) + C \,.$$

Hence, by using the substitution $t = z_{0j}r$, we get

$$\begin{aligned}
\int_0^1 r(1 - r^2) J_0(z_{0j}r) dr &= \frac{1}{z_{0j}^4} \int_0^{z_{0j}} \left[z_{0j}^2 t - t^3 \right] J_0(t) dt \\
&= \frac{1}{z_{0j}^4} \left[(z_{0j}^2 t - t^3) J_1(t) + 2t^2 J_2(t) \right]_0^{z_{0j}} \\
&= \frac{2 J_2(z_{0j})}{z_{0j}^2}
\end{aligned}$$

and so

$$A_{0j} = \frac{4 J_2(z_{0j})}{z_{0j}^2 J_1(z_{0j})^2} \,.$$

The solution u is

$$u(r, \theta, t) = 4 \sum_{j=1}^\infty \frac{J_2(z_{0j}) J_0(z_{0j}r)}{z_{0j}^2 J_1(z_{0j})^2} \cos(2z_{0j}t) + \sum_{j=1}^\infty B_{1j} J_1(z_{1j}r) \cos(2z_{1j}t) \sin \theta \,.$$

5. HEAT CONDUCTION IN A CIRCULAR PLATE

The following boundary value problem models heat propagation in a thin circular plate with insulated faces and with heat transfer on the circular boundary. In polar coordinates, the problem is:

(9) $$\begin{cases} u_t = k \left(u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2} \right) & r < L, \ 0 \leq \theta \leq 2\pi, \ t > 0, \\ hu(L, \theta, t) + Lu_r(L, \theta, t) = 0 & 0 \leq \theta \leq 2\pi, \ t > 0, \\ u(r, \theta, 0) = f(r, \theta) & r < L, \ 0 \leq \theta \leq 2\pi, \end{cases}$$

where $h \geq 0$ is a constant. Let us assume for simplicity that f is rotation invariant $f = f(r)$ is independent on θ. We seek then a solution that is also independent on θ and the problem becomes.

$$(10) \qquad \begin{cases} u_t = k \left(u_{rr} + \dfrac{u_r}{r} \right) & r < L, \ t > 0, \\ hu(L,t) + Lu_r(L,t) = 0 & t > 0, \\ u(r) = f(r) & r < L. \end{cases}$$

The solutions with separated variables: $u(r,t) = R(r)T(t)$ of the homogeneous part leads to the ODE problems

$$\begin{cases} r^2 R'' + r R' + \lambda r^2 R = 0 \\ hR(L) + LR'(L) = 0 \end{cases} \quad \text{and} \quad T' + (k\lambda)T = 0 \ .$$

The R-equation is the parametric Bessel equation of order 0. The values $\lambda < 0$ cannot be eigenvalues. The case $\lambda = 0$ is a particular case.

Remark about the case $\lambda = 0$. In this case, the R-equation is also a Cauchy-Euler equation with general solution $A + B \ln r$ and the bounded solutions are $R(r) = A$. Such a solutions satisfies the boundary condition if and only if $h = 0$. This correspond to the situation where the boundary is insulated

The (positive) eigenvalues and eigenfunctions are:

$$\lambda_j = \frac{z_j^2}{L^2}, \qquad R_j(r) = J_0 \left(\frac{z_j r}{L} \right), \qquad j \in \mathbb{Z}^+,$$

where z_j is the j-th positive root of the equation

$$hJ_0(z) + zJ_0'(z) = 0 \ .$$

A corresponding T-solution is

$$T_j(t) = \exp \left(-\frac{kz_j^2}{L^2} t \right) \ .$$

The series solution of the homogeneous part, when $h > 0$, is

$$u(r,t) = \sum_{j=1}^{\infty} C_j e^{-(kz_j^2/L^2)t} J_0 \left(\frac{z_j r}{L} \right)$$

and when $h = 0$ is

$$u(r,t) = C_0 + \sum_{j=1}^{\infty} C_j e^{-(kz_j^2/L^2)t} J_0 \left(\frac{z_j r}{L} \right)$$

In order for such a series to satisfy the nonhomogeneous condition, we need

$$u(r,0) = f(r) = C_0 + \sum_{j=1}^{\infty} C_j J_0 \left(\frac{z_j r}{L} \right)$$

(with $C_0 = 0$ when $h > 0$). This is the J_0-Bessel series of f with endpoint condition $hJ_0(z) + zJ_0'(z) = 0$. We have then (see Theorem 2),

$$C_j = \frac{1}{||J_0(z_j r/L)||_r^2} \int_0^L r f(r) J_0 \left(\frac{z_j r}{L} \right) dr = \frac{2z_j^2}{L^2(z_j^2 + h^2)J_0(z_j)^2} \int_0^L r f(r) J_0 \left(\frac{z_j r}{L} \right) dr \ .$$

Remark. If instead of being independent on θ, the initial condition is of the form $f(r)\cos(m\theta)$, the solution u of problem (9) can be put in the form

$$u(r, \theta, t) = v(r, t)\cos(m\theta),$$

where v solves the two variables problem

(11)
$$\begin{cases} v_t = k\left(v_{rr} + \dfrac{v_r}{r} - m^2\dfrac{v}{r^2}\right) & r < L, \ t > 0, \\ hv(L, t) + Lv_r(L, t) = 0 & t > 0, \\ v(r) = f(r) & r < L. \end{cases}$$

This time the separation of variables of the v-problem leads to the solution v of the form

$$v(r, t) = \sum_{j=1}^{\infty} C_j e^{-(kz_j^2/L^2)t} J_m\left(\frac{z_j r}{L}\right)$$

6. Helmholtz Equation in the Disk

The Helmholtz equation in the disk is the following two-dimensional eigenvalue problem: Find λ and nontrivial functions $u(r, \theta)$ defined in the disk of radius L such that:

(12)
$$\begin{cases} \Delta u(r, \theta) + \lambda u(r, \theta) = 0 & 0 < r < L, \ 0 \le \theta \le 2\pi, \\ u(L, \theta) = 0 & 0 \le \theta \le 2\pi. \end{cases}$$

The solutions with separated variables $u(r, \theta) = R(r)\Theta(\theta)$ leads to the following one-dimensional eigenvalue problems

$$\begin{cases} \Theta''(t) + \alpha^2\Theta(t) = 0 \\ \Theta \text{ and } \Theta' \ 2\pi - \text{periodic} \end{cases}$$

and

$$\begin{cases} r^2 R''(r) + rR'(r) + (\lambda r^2 - \alpha^2)R(r) = 0 \\ R(L) = 0. \end{cases}$$

The eigenvalues and eigenfunctions of the Θ-problem are

$$\alpha_m^2 = m^2, \quad \Theta_m^1(\theta) = \cos(m\theta) \ \text{ and } \ \Theta_m^2(\theta) = \sin(m\theta) \quad m = 0, \ 1, \ 2, \cdots$$

For each m, the R-problem is an order m Bessel type eigenvalues problem with eigenvalues and eigenfunctions

$$\lambda_{j,m} = \left(\frac{z_{j,m}}{L}\right)^2, \qquad R_{j,m}(r) = J_m\left(\frac{z_{j,m}r}{L}\right).$$

where $z_{j,m}$ is the j-th positive roots of the equation $J_m(z) = 0$.

The eigenvalues and eigenfunctions of the Helmholtz equation are

$$\lambda_{j,m} = \frac{z_{j,m}^2}{L^2}, \qquad \begin{cases} u_{j,m}^1(r, \theta) = \cos(m\theta)J_m\left(\dfrac{z_{j,m}r}{L}\right), \\ u_{j,m}^2(r, \theta) = \sin(m\theta)J_m\left(\dfrac{z_{j,m}r}{L}\right), \end{cases}$$

Eigenfunctions expansion for a nonhomogeneous problem. The eigenfunctions $u_{j,m}^1$, $u_{j,m}^2$ can can be used to solve nonhomogeneous problems such as the following

(13)
$$\begin{cases} \Delta u(r, \theta) + au(r, \theta) = F(r, \theta) & 0 < r < L, \ 0 \le \theta \le 2\pi, \\ u(L, \theta) = 0 & 0 \le \theta \le 2\pi. \end{cases}$$

where a is a constant. We expand F and u into these eigenfunctions (Fourier-Bessel series). We seek then a solution u of the form

$$u(r, \theta) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} [A_{m,j} \cos(m\theta) + B_{m,j} \sin(m\theta)] J_m \left(\frac{z_{j,m} r}{L} \right)$$

By using $\Delta u_{m,j}^{1,2} = -\lambda_{m,j} u_{m,j}^{1,2}$, we deduce that

$$(14) \quad \Delta u + au = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} (a - \lambda_{m,j}) [A_{m,j} \cos(m\theta) + B_{m,j} \sin(m\theta)] J_m \left(\frac{z_{j,m} r}{L} \right)$$

The Fourier-Bessel series of F is

$$(15) \quad F(r, \theta) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \left[F_{m,j}^1 \cos(m\theta) + F_{m,j}^2 \sin(m\theta) \right] J_m \left(\frac{z_{j,m} r}{L} \right)$$

where the coefficients $F_{m,j}^{1,2}$ are given by
(16)
$$\left. \begin{matrix} F_{m,j}^1 \\ F_{m,j}^2 \end{matrix} \right\} = \frac{1}{\pi ||J_m(z_{m,j}r/L)||_r^2} \int_0^{2\pi} \int_0^L F(r,\theta) r J_m \left(\frac{z_{m,j}r}{L} \right) dr \left\{ \begin{matrix} \cos(m\theta) \\ \sin(m\theta) \end{matrix} \right. d\theta .$$

By identifying the coefficients in the series (14) and (15), we get

$$A_{m,j} = \frac{F_{m,j}^1}{a - \lambda_{m,j}} \quad \text{and} \quad B_{m,j} = \frac{F_{m,j}^2}{a - \lambda_{m,j}} .$$

provided that $a \neq \lambda_{m,j}$ for every m, j (nonresonant case).

Example. Consider the problem

$$\begin{cases} \Delta u(r, \theta) - u(r, \theta) = 1 + r^2 \sin(2\theta) & 0 < r < 3, \ \ 0 \leq \theta \leq 2\pi , \\ u(3, \theta) = 0 & 0 \leq \theta \leq 2\pi . \end{cases}$$

In this problem, we have $F_{m,j}^{1,2} = 0$, for every m except when $m = 0$ and $m = 2$. Furthermore, $F_{2,j}^1 = 0$ for every j. For the remainder of the coefficients, we have

$$\begin{aligned} F_{0,j}^1 &= \frac{1}{||J_0(z_{0,j}r/3)||_r^2} \int_0^3 r J_0(z_{0,j}r/3) dr \\ &= \frac{2}{z_{0,j}^2 J_1(z_{0,j})^2} \int_0^{z_{0,j}} t J_0(t) dt \\ &= \frac{2}{z_{0,j} J_1(z_{0,j})} \end{aligned}$$

and

$$\begin{aligned} F_{2,j}^2 &= \frac{1}{||J_2(z_{2,j}r/3)||_r^2} \int_0^3 r^3 J_2(z_{0,j}r/3) dr \\ &= \frac{2(9^2)}{z_{2,j}^4 J_3(z_{2,j})^2} \int_0^{z_{2,j}} t^3 J_0(t) dt \\ &= \frac{162}{z_{2,j} J_3(z_{2,j})} \end{aligned}$$

The corresponding coefficients of the solutions are

$$A_{0,j}^1 = \frac{F_{0,j}^1}{-1 - (z_{0,j}^2/9)} = -\frac{18}{z_{0,j}(9 + z_{0,j}^2)J_1(z_{0,j})}$$

$$B_{2,j}^1 = \frac{F_{2,j}^2}{-1 - (z_{2,j}^2/9)} = -\frac{1458}{z_{2,j}(9 + z_{2,j}^2)J_3(z_{2,j})}$$

The solution of the problem is

$$u = -18 \sum_{j=1}^{\infty} \frac{J_0(z_{0,j}r/3)}{z_{0,j}(9 + z_{0,j}^2)J_1(z_{0,j})} - 1458 \sum_{j=1}^{\infty} \frac{J_2(z_{2,j}r/3)\sin(2\theta)}{z_{2,j}(9 + z_{2,j}^2)J_3(z_{2,j})}$$

7. EXERCISES

In Exercises 1 to 9, find the J_α-Bessel series of the given function $f(x)$ over the given interval and with respect to the given endpoint condition.

Exercise 1. $f(x) = 100$ over $[0,\ 5]$, and $J_0(z) = 0$.

Exercise 2. $f(x) = x$ over $[0,\ 7]$, and $J_1(z) = 0$.

Exercise 3. $f(x) = -5$ over $[0,\ 1]$, and $J_2(z) = 0$.

Exercise 4. $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases}$ over $[0,\ 2]$, and $J_0(z) = 0$.

Exercise 5. $f(x) = x$ over $[0,\ 3]$, and $J_1'(z) = 0$

Exercise 6. $f(x) = 1$ over $[0,\ 3]$, and $J_0(z) + zJ_0'(z) = 0$.

Exercise 7. $f(x) = x^2$ over $[0,\ 3]$, and $J_0(z) = 0$ (leave the coefficients in an integral form).

Exercise 8. $f(x) = x^2$ over $[0,\ 3]$, and $J_3(z) = 0$ (leave the coefficients in an integral form).

Exercise 9. $f(x) = \sqrt{x}$ over $[0,\ \pi]$, and $J_{1/2}(z) = 0$ (use the explicit expression of $J_{1/2}$ and relate to Fourier series).

In exercises 11 to 14, solve the indicated boundary value problem that deal with heat flow in a circular domain.

Exercise 11.
$$
\begin{aligned}
u_t &= 2\left(u_{rr} + \frac{u_r}{r}\right) & & 0 < r < 2,\ t > 0, \\
u(2,t) &= 0 & & t > 0, \\
u(r,0) &= 5 & & 0 < r < 2.
\end{aligned}
$$

Exercise 12.
$$
\begin{aligned}
u_t &= 2\left(u_{rr} + \frac{u_r}{r}\right) & & 0 < r < 2,\ t > 0, \\
u_r(2,t) &= 0 & & t > 0, \\
u(r,0) &= 5 & & 0 < r < 2.
\end{aligned}
$$

Exercise 13.
$$
\begin{aligned}
u_t &= 2\left(u_{rr} + \frac{u_r}{r}\right) & & 0 < r < 2,\ t > 0, \\
2u(2,t) - u_r(2,t) &= 0 & & t > 0, \\
u(r,0) &= 5 & & 0 < r < 2.
\end{aligned}
$$

Exercise 13. Find a solution u of the form $u(r, \theta, t) = v(r, t) \sin(2\theta)$ of the problem.

$$u_t = 2 \left(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \right) \qquad 0 < r < 2, \ 0 \leq \theta \leq 2\pi, \ t > 0,$$
$$u(2, \theta, t) = 0 \qquad 0 \leq \theta \leq 2\pi, \ t > 0,$$
$$u(r, \theta, 0) = 5r^2 \sin(2\theta) \qquad 0 < r < 2, \ 0 \leq \theta \leq 2\pi, \ .$$

Exercise 14.

$$u_t = \left(u_{rr} + \frac{u_r}{r} - \frac{9u}{r^2} \right) \qquad 0 < r < 1, \ t > 0,$$
$$u(1, t) = 0 \qquad t > 0,$$
$$u(r, 0) = r^3 \qquad 0 < r < 1.$$

In exercises 15 to 18, solve the indicated boundary value problem that deal with wave propagation in a circular domain.

Exercise 15.

$$u_{tt} = 2 \left(u_{rr} + \frac{u_r}{r} \right) \qquad 0 < r < 3, \ t > 0,$$
$$u(3, t) = 0 \qquad t > 0,$$
$$u(r, 0) = 9 - r^2 \qquad 0 < r < 3,$$
$$u_t(r, 0) = 0 \qquad 0 < r < 3.$$

Exercise 16.

$$u_{tt} = \left(u_{rr} + \frac{u_r}{r} \right) \qquad 0 < r < 1, \ t > 0,$$
$$u(1, t) = 0 \qquad t > 0,$$
$$u(r, 0) = 0 \qquad 0 < r < 1,$$
$$u_t(r, 0) = g(r) \qquad 0 < r < 1.$$

where

$$g(r) = \begin{cases} 1 & \text{if } 0 < r < 1/2, \\ 0 & \text{if } (1/2) < r < 1. \end{cases}$$

Exercise 17. Find a solution u of the form $u(r, \theta, t) = v(r, t) \cos \theta$

$$u_{tt} = \left(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \right) \qquad 0 < r < 1, \ t > 0,$$
$$u(1, t) = 0 \qquad t > 0,$$
$$u(r, 0) = 0 \qquad 0 < r < 1,$$
$$u_t(r, 0) = g(r) \cos \theta \qquad 0 < r < 1.$$

where

$$g(r) = \begin{cases} r & \text{if } 0 < r < 1/2, \\ 0 & \text{if } (1/2) < r < 1. \end{cases}$$

Exercise 18. Find a solution u of the form $u(r, \theta, t) = v(r, t) \cos \theta$

$$u_{tt} = \left(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \right) \qquad 0 < r < 1, \ t > 0,$$
$$u(1, t) = 0 \qquad t > 0,$$
$$u(r, 0) = J_1(z_{1,1}r) \sin \theta \qquad 0 < r < 1,$$
$$u_t(r, 0) = g(r) \cos \theta \qquad 0 < r < 1.$$

where $z_{1,1}$ is the first positive zero of J_1 and where

$$g(r) = \begin{cases} r & \text{if } 0 < r < 1/2, \\ 0 & \text{if } (1/2) < r < 1. \end{cases}$$

In exercises 19 to 21, solve the Helmholtz equation in the disk with radius L

Exercise 19. $L = 2$

$$\Delta u - u = 2, \quad u(2, \theta) = 0$$

Exercise 20. $L = 1$

$$\Delta u = r \sin \theta, \quad u(1, \theta) = 0$$

Exercise 21. $L = 3$

$$\Delta u + 2u = -1 + 5r^3 \cos(3\theta), \quad u(3, \theta) = 0$$

Exercise 22. Solve the following Dirichlet problem in the cylinder with radius 10 and height 20

$$
\begin{aligned}
u_{rr} + \frac{u_r}{r} + u_{zz} &= 0 && 0 < r < 10, \ 0 < z < 20, \\
u(10, z) &= 0 && 0 < z < 20, \\
u(r, 0) &= 0 && 0 < r < 10, \\
u(r, 20) &= 1 && 0 < r < 10.
\end{aligned}
$$