THE WAVE EQUATION

The aim is to derive a mathematical model that describes small vibrations of a tightly stretched flexible string for the one-dimensional case, or of a tightly stretched membrane for the dimensional case. The derivation of these models is mainly based on *Newton's Second Law of Motion* (Force = mass × acceleration).

1. The One-Dimensional Wave Equation

We derive the simplest form of the wave equation for the idealized string by making a number of assumption on the physical string.

Assume that a flexible string of length L is tightly stretched along the x-axis so that one of its end point is at $x = 0$ and the other at $x = L$. We assume that the tension force on the string is the dominant force and all other forces acting on the string are negligible (no external forces are applied to the string, the damping forces (resistance) are negligible, and that the weight of the string is also negligible). The flexibility of the string implies that at each point, the tension force has constant magnitude and has the direction of the tangent line to the string. We also assume that each point of the string moves only vertically[1]

Let $u(x,t)$ denotes the (vertical) displacement at time t of the point x on the string. Note that at a fixed time $t = t_0$, the shape of the string is given by graph the function $u(x,t_0)$ (See Figure 1.). Our aim is to find an equation that is satisfied by $u(x,t)$. Consider a small element of the string between the points x and $x + \Delta x$ ($\Delta x > 0$ small and we are assuming that this element moves vertically). The total force to which this element is subject to is the tension force exerted at the left end $\vec{T}(x,t)$ and the tension force exerted at the right end $\vec{T}(x+\Delta x,t)$ by the rest of the string. These forces have the same constant magnitude T:

$$|\vec{T}(x,t)| = |\vec{T}(x+\Delta x,t)| = T, \qquad \forall x, \forall t.$$

Let $\theta(x,t)$ be the angle between $\vec{T}(x,t)$ and $\vec{i}$ and $\theta(x+\Delta x,t)$ be the angle between $\vec{T}(x+\Delta x,t)$ and $\vec{i}$. We take these angles to be between 0 and π. Since we are assuming that we are dealing with small vibrations then either θ is close to 0 or close to π. In Figure 2, $\theta(x,t)$ is close is close to π and $\theta(x+\Delta x,t)$ is close is close to 0 (for illustration purposes, Figure 2. is out of proportion and does not convey that the vibrations are small).

The total vertical force acting on the element is

$$\begin{aligned} F &= \vec{T}(x,t)\cdot\vec{j} + \vec{T}(x+\Delta x,t)\cdot\vec{j} \\ &= T\left(\sin(\theta(x,t)) + \sin(\theta(x+\Delta x,t))\right) \end{aligned} \tag{1}$$

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[1]In fact a given point on the string moves up and down not along the vertical but along a curve that is slightly deflected from the vertical vertical. But the horizontal displacement is very small compared to the vertical displacement. This is why we are justified in assuming only vertical displacements.

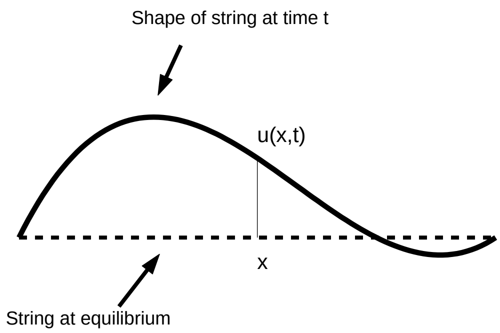

FIGURE 1. *Shape the string at fixed time t given by the graph of $f(x) = u(x,t)$*

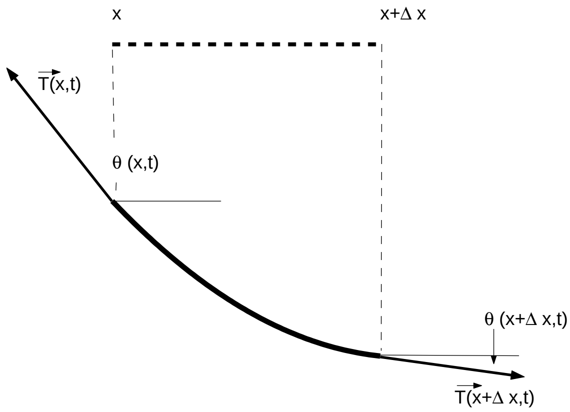

FIGURE 2. *Element of string at time t subject to tension forces*

Recall the following linear approximation formulas

$$\sin\theta \approx \theta \approx \tan\theta \qquad \text{for } \theta \text{ close to } 0.$$

and

$$\sin\theta \approx \pi - \theta \approx -\tan\theta \qquad \text{for } \theta \text{ close to } \pi.$$

Recall also that the slope of the tangent line to the graph of a function $y = f(x)$, through the point (x_0, y_0), is

$$\frac{df}{dx}(x_0) = \tan\theta_0\,,$$

where θ_0 is the inclination angle. In our case the shape of the string at a fixed time t is given as the graph of the function $u(x,t)$ (t fixed and x varies). We have then

$$\tan\theta(x,t) = \frac{\partial u}{\partial x}(x,t) \quad \text{and} \quad \tan\theta(x+\Delta x,t) = \frac{\partial u}{\partial x}(x+\Delta x,t) \tag{2}$$

By using (1), (2), and the approximation formulas for $\sin\theta$, we obtain the following approximation for the vertical force acting on the element of the string

$$F \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right) \tag{3}$$

Now, we use Newton's second law to replace F in (3) by $\Delta m \cdot a$, where a is the acceleration of the element at time t and Δm is the mass of the element to obtain

$$\Delta m \cdot a \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right) \tag{4}$$

But the acceleration is the second derivative (w.r.t time) of the position function and the mass is the linear density ρ of the string times the length:

$$\Delta m \approx \rho \Delta x \quad \text{and} \quad a \approx \frac{\partial^2 u}{\partial t^2}(x,t).$$

Hence, after replacing a and Δm by these expressions in (4) and dividing by Δx, we obtain

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) \approx T \frac{\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x} \tag{5}$$

The approximations in (5) become better and better as Δx becomes smaller and smaller. At the limit ($\Delta x \to 0$), we obtain

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) = T \frac{\partial^2 u}{\partial x^2}(x,t). \tag{6}$$

Finally, we divide in (6) by ρ and set $c = \sqrt{\frac{T}{\rho}}$ to rewrite the equation as

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t). \tag{7}$$

Equation (7) is known as the *one-dimensional wave equation.* The units of the constant c are

$$\begin{aligned} [c] &= ([T]/[\rho])^{1/2} = (\text{Force}/\text{Density})^{1/2} \\ &= \left(\frac{(\text{Mass})\cdot\ (\text{Length}/\text{Time}^2)}{\text{Mass}/\text{Length}}\right)^{1/2} = \frac{\text{Length}}{\text{Time}} \end{aligned}$$

Hence c has the units of velocity and it is called the *wave's speed*.

Remark 1. We can interpret c as the speed with which the crest of the wave travels horizontally (the points on the string travel vertically and their velocity is not c).

Remark 2. Assume that the string is not homogeneous so that the density ρ and the magnitude of tension force T depend on x. In this case the wave equation takes the form

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left(T(x) \frac{\partial u}{\partial x}(x,t)\right)$$

or equivalently

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c(x)^2 \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{T'(x)}{\rho(x)} \frac{\partial u}{\partial x}(x,t).$$

Remark 3. If for the homogeneous string the damping forces are not neglected, then the new wave equation might take the form

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \alpha \frac{\partial u}{\partial t}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)\,.$$

Remark 4. If the homogeneous string is subject to an external force, then the standard wave equation (7) becomes

$$\frac{\partial^2 u}{\partial t^2}(x,t) \;=\; c^2\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t)\,.$$

Remark 5. The general 1-dimensional wave equation in which many parameters are not neglected has the form

$$\frac{\partial^2 u}{\partial t^2}(x,t) \;=\; c^2\frac{\partial^2 u}{\partial x^2}(x,t) + \alpha\frac{\partial u}{\partial x}(x,t) + \beta\frac{\partial u}{\partial t}(x,t) + \gamma u(x,t) + F(x,t)\,,$$

in which the coefficients might depend on x and t.

2. The Two-Dimensional Wave Equation

Consider a homogeneous, well stretched, and flexible membrane whose boundary is held fixed in the (x,y)-plane. As with the case of the string, we assume that the tension force is the dominant force governing the vibrations. We also assume that the vibrations vertical and small compared to the size of the membrane. Our aim is to construct a model for the vertical displacement function.

Let $u(x,y,t)$ represents the vertical displacement at time t of the point (x,y) of the membrane. Consider a small rectangular element of the membrane $ABCD$ that corresponds to the points (x,y), $(x+\Delta x,y)$, $(x+\Delta x,y+\Delta y)$, and $(x,y+\Delta y)$. We assume that this portion of the membrane has a vertical motion under the action of the tension forces (see Figure 3). As in the case of the string, the flexibility of

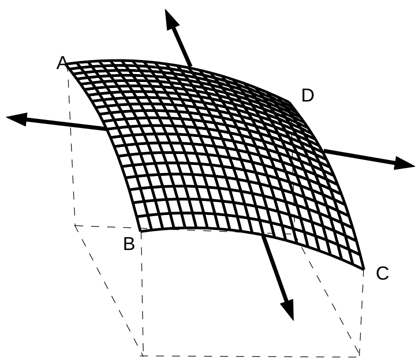

Figure 3. *Element $ABCD$ of the membrane at time t subject to tension forces on the edges*

the membrane implies that the tension $\vec{T}(x,y,t)$ has constant magnitude T at each point of the membrane and that $\vec{T}$ is tangent to the surface. Since we are dealing with small vibrations, then we can assume that at each time t and at each point the angle θ between the the normal to the surface and the z-axis is close to 0 (which is the same as the angle between the tangent space and the horizontal plane). By using the linear approximation $\sin\theta \approx \tan\theta$, similar arguments as those used for the string, imply that the total (vertical force) F acting on the element $ABCD$ can be approximated as follows:

$$\begin{aligned} F \;&\approx\; T\left[\Delta y\left(\frac{\partial u}{\partial x}(x+\Delta x,y,t)-\frac{\partial u}{\partial x}(x,y,t)\right)+\right.\\ &\qquad\left.+\,\Delta x\left(\frac{\partial u}{\partial y}(x,y+\Delta y,t)-\frac{\partial u}{\partial y}(x,y,t)\right)\right] \end{aligned} \tag{8}$$

We can also use Newton's second law to express F as

$$F \approx \Delta m\,\frac{\partial^2 u}{\partial t^2}(x,y,t) \approx \rho\Delta x\,\Delta y\,\frac{\partial^2 u}{\partial t^2}(x,y,t), \tag{9}$$

where $\Delta m \approx \rho\Delta x\,\Delta y$ denotes the mass of the rectangular element $ABCD$ and with ρ the surface mass density of the membrane. After substituting in (8), the expression for F given in (9) and dividing by $\Delta x\Delta y$, we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,y,t) \;&\approx\; T\left[\frac{\frac{\partial u}{\partial x}(x+\Delta x,y,t)-\frac{\partial u}{\partial x}(x,y,t)}{\Delta x}+\right.\\ &\qquad\left.+\,\frac{\frac{\partial u}{\partial y}(x,y+\Delta y,t)-\frac{\partial u}{\partial y}(x,y,t)}{\Delta y}\right] \end{aligned} \tag{10}$$

The approximations in (10) become better and better as Δx and Δy become smaller and smaller. At the limit we obtain the *two-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2}(x,y,t) = c^2\left(\frac{\partial^2 u}{\partial x^2}(x,y,t)+\frac{\partial^2 u}{\partial y^2}(x,y,t)\right) \tag{11}$$

where $c^2 = T/\rho$ is the wave's speed.

3. The Three-Dimensional Wave Equation

With the use of the notation Δ for the Laplace operator, the wave in equation in one, two, or three space variables takes the form

$$u_{tt} = c^2\Delta u\,.$$

We have illustrated the wave equation in connection with the vibrations of the string and of the membrane. But the equation models many other physical phenomena that includes acoustic waves, electrical circuits, quantum mechanics in connection with the Shrödinger's equation etc.

4. Some Solutions Of The Wave Equation

The followings are solutions of the one-dimensional wave equation $u_{tt} = c^2u_{xx}$.

$$\sin(c\omega_1 t)\sin(\omega_1 x) \quad \text{and} \quad \cos(c\omega_2 t)\sin(\omega_2 x),$$

where ω_1 and ω_2 are arbitrary real constants. We can interchange any sin with cos and obtain again a solution. We can multiply any solution by a constant or add any two solutions and the result is again a solution of the wave equation (verify these claims).

For the two dimensional wave equation $u_{tt} = c^2 \Delta u$, here are some solutions

$$\sin(c\sqrt{\omega_1^2 + \omega_2^2}\, t) \sin(\omega_1 x) \cos(\omega_2 y)$$

with $\omega_1,\ \omega_2$ constants. Again we can produce more solutions by interchanging any sine function with cosine and vice versa and by adding or multiplying solutions by constants.

5. Boundary Value Problems For The Wave Equation

We list some typical BVP dealing with the wave equation.

5.1. **Example 1.** A flexible string of length L is stretched horizontally so that one end is at $x = 0$ and the other at $x = L$. While the ends are held fixed, the string is moved vertically from equilibrium so that the point x is displaced by $f(x)$ (with $f(x) > 0$ or $f(x) < 0$ depending whether the point x is moved above or below its equilibrium position). Hence the shape of the string is given by the graph of f. The string is then released from *rest* (meaning that when the string is released each point of the string has 0 velocity). Let $u(x, t)$ denotes the (vertical) displacement

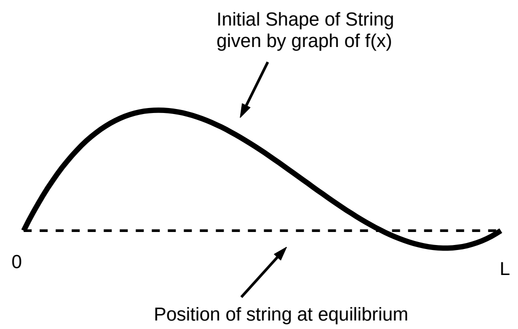

Figure 4. *String displaced from equilibrium and released from rest*

at time t of point x. This function satisfies the followings:

- the PDE (wave equation): $u_{tt}(x, t) = c^2 u_{xx}(x, t)$
- the initial position: $u(x, 0) = f(x)$
- the initial velocity: $u_t(x, 0) = 0$
- the left ends are fixed: $u(0, t) = 0$ and $u(L, t) = 0$

The formal BVP for u can be written as:

$$\begin{array}{ll}
u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, \quad t > 0; \\
u(x,0) = f(x) & 0 < x < L; \\
u_t(x,0) = 0 & 0 < x < L; \\
u(0,t) = 0 & t > 0; \\
u(L,t) = 0 & t > 0;
\end{array}$$

5.2. **Example 2.** Start with the string of Example 1 at equilibrium (sitting horizontally). Suppose that at the initial time each point x of the string is given a velocity $g(x)$ (with $g(x)$ positive or negative depending whether the velocity is directed up or down). This time the displacement $u(x, t)$ satisfies the followings:

- the PDE (wave equation): $u_{tt}(x, t) = c^2 u_{xx}(x, t)$

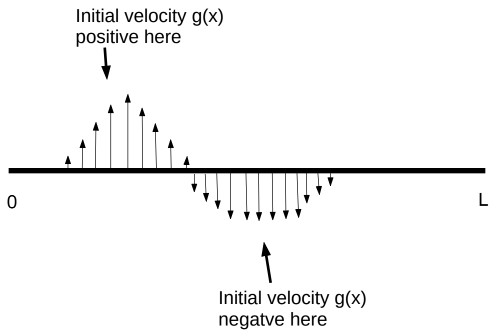

FIGURE 5. *String struck from equilibrium with an initial velocity $g(x)$*

- the initial position: $u(x, 0) = 0$
- the initial velocity: $u_t(x, 0) = g(x)$
- the left ends are fixed: $u(0, t) = 0$ and $u(L, t) = 0$

The formal BVP for u can be written as:

$$\begin{array}{ll} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, \quad t > 0; \\ u(x,0) = 0 & 0 < x < L; \\ u_t(x,0) = g(x) & 0 < x < L; \\ u(0,t) = 0 & t > 0; \\ u(L,t) = 0 & t > 0; \end{array}$$

5.3. **Example 3.** The combination of both previous experiments, the string is initially moved from the equilibrium position by $f(x)$ and each point is given an initial velocity $g(x)$ The formal BVP for the displacement u is:

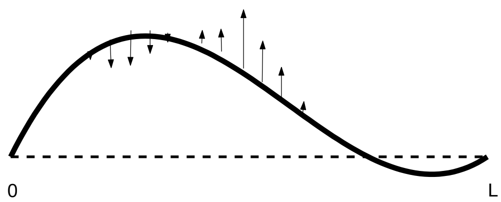

FIGURE 6. *String with initial position $f(x)$ and initial velocity $g(x)$*

$$\begin{array}{ll} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, \quad t > 0; \\ u(x,0) = f(x) & 0 < x < L; \\ u_t(x,0) = g(x) & 0 < x < L; \\ u(0,t) = 0 & t > 0; \\ u(L,t) = 0 & t > 0; \end{array}$$

5.4. **Example 4.** Suppose this time that the right end point of the string is not held fixed but is allowed to move vertically in such a way that the shape of the string stays always horizontal at the end $x = L$. This can be realized by attaching the end of the string to a mechanical apparatus (that contain bearings or springs). Now the displacement $u(L, t)$ is not zero anymore. But, the condition that the right end stick out from the mechanical device in the horizontal direction means that the

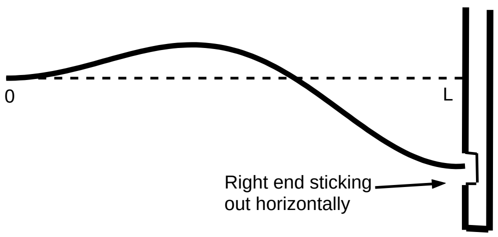

FIGURE 7. *String with left end fixed and right end attached to a mechanical device that allows it to move vertically*

slope of the tangent line to the string is 0 at $x = L$. That is $u_x(L,t) = 0$ for all time $t > 0$. The BVP for u becomes

$$\begin{array}{lll} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, & t > 0; \\ u(x,0) = f(x) & 0 < x < L; & \\ u_t(x,0) = g(x) & 0 < x < L; & \\ u(0,t) = 0 & t > 0; & \\ u_x(L,t) = 0 & t > 0; & \end{array}$$

5.5. **Example 5.** Consider a 2×1 rectangular membrane. Suppose that the membrane is flexible and is well stretched on the (x,y)-plane so that it occupies the rectangle $[0,\ 2] \times [0,\ 1]$. Assume that its boundary is held fixed throughout. At time $t = 0$ the membrane is displaced from its equilibrium position so that each point (x,y) is displaced vertically by the amount $xy(x-1)(2-x)(y-1)$ and then it is released from rest. Let $u(x,y,t)$ denotes the vertical displacement at time t of

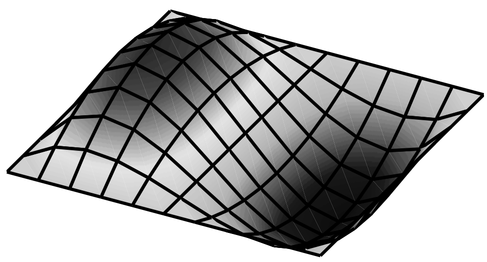

FIGURE 8. *A rectangular membrane displaced from its equilibrium position*

the point (x,y). The function u satisfies:

- the PDE (2-D wave equation): $u_{tt} = c^2(u_{xx} + u_{yy})$
- the initial position: $u(x,y,0) = xy(x-1)(2-x)(y-1)$
- the initial velocity: $u_t(x,y,0) = 0$
- the boundary is fixed: $u(0,y,t) = u(2,y,t) = u(x,0,t) = u(x,1,t) = 0$

As a formal BVP, we can write the problem for u as

$$\begin{array}{ll} u_{tt}(x,t) = c^2\Delta u(x,y,t) & 0<x<2,\ 0<y<1,\ t>0; \\ u(x,y,0) = xy(x-1)(2-x)(y-1) & 0<x<2,\ 0<y<1 \\ u_t(x,y,0) = 0 & 0<x<2,\ 0<y<1 \\ u(0,y,t) = u(2,y,) = 0 & 0<y<1,\ t>0; \\ u(x,0,t) = u(x,1,t) = 0 & 0<x<2,\ t>0; \end{array}$$

6. The D'Alembert's Solution Of The Wave Equation

We start with an observation, let $F(s)$ be a twice continuously differentiable function of one real variable ($s \in \mathbb{R}$). Form the function F, form the functions of two variables $u^+(x,t)$ and $u^-(x,t)$ defined in $\mathbb{R}^2$ by

$$u^+(x,t) = F(x-ct) \quad \text{and} \quad u^-(x,t) = F(x+ct).$$

Lemma *The functions u^+ and u^- are solutions of the wave equation*

$$u_{tt}(x,t) = c^2 u_{xx}(x,t).$$

Proof. By using the chain rule, we have

$$u_t^+(x,t) = -cF'(x-ct), \qquad u_x^+(x,t) = F'(x-ct)$$

and

$$u_{tt}^+(x,t) = (-c)^2F''(x-ct), \qquad u_{xx}^+(x,t) = F''(x-ct)$$

From these it follows that $u_{tt}^+ = c^2 u_{xx}^+$. The same argument applies to u^-.

6.1. **The string with infinite length.** Consider a long string (for practical purposes, we can assume that the string has an infinite length so that for now we will not worry about the end points). Suppose that the string is perturbed from its (horizontal) equilibrium position so that its new shape is that of the graph of the function F (see figure). Then, at time $t = 0$, the string is released from rest from this position. We would like to find the shape of the string at each time $t > 0$. Hence we would like to find the solution of the following

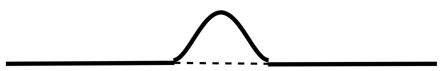

Figure 9. *A long string with a small perturbation*

Problem. Solve the initial value problem

$$\begin{array}{ll} u_{tt}(x,t) = c^2u_{xx}(x,t) & x\in\mathbb{R},\ t>0 \\ u(x,0) = F(x) \quad u_t(x,0) = 0 & x \in \mathbb{R}. \end{array}$$

Note that both functions $u^\pm$ solve the PDE and the initial position

$$u^+(x,0) = u^-(x,0) = F(x \pm 0) = F(x).$$

However, they do not satisfy the initial velocity since

$$u_t^+(x,t) = -cF'(x-ct), \qquad u_t^-(x,t) = cF'(x+ct)$$

and at $t = 0$, we have

$$u_t^+(x,0) = -cF'(x), \qquad u_t^-(x,0) = cF'(x)$$

which are not 0 (unless F is constant). To circumvent this, we use the fact that a linear combination of two solutions of the wave equation is again a solution. The average u of u^+ and u^- is again a solution:

$$u(x,t) = \frac{u^+(x,t) + u^-(x,t)}{2}.$$

It clearly satisfies the initial position and the initial velocity since

$$u_t(x,0) = \frac{u_t^+(x,0) + u_t^-(x,0)}{2} = \frac{-cF'(x) + cF'(x)}{2} = 0.$$

We have therefore found the solution of the initial value problem as

$$u(x,t) = \frac{F(x-ct)}{2} + \frac{F(x+ct)}{2}.$$

The function u can be interpreted as the sum of the two waves $u^+/2$ traveling to the right with speed c and $u^-/2$ traveling to the left with speed c. Indeed at a any time $t = t_0$, the graph of $u^+(x,t_0)$ is obtained from that of F by a translation to the right of ct_0 units while that of $u^-(x,t_0)$ is just the translation of that of F by ct_0 units to the left.

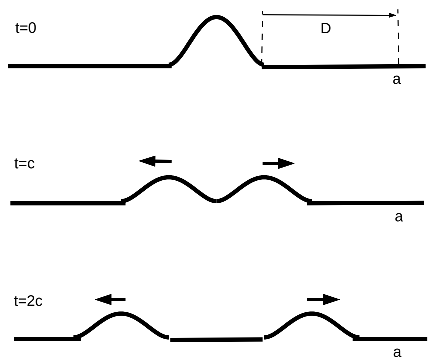

FIGURE 10. *The perturbation F of the string spreads for $t > 0$ as two traveling waves $u^+/2$ and $u^-/2$ with speed c*

Remark. Consider a point a on the string so that $F(s) = 0$ for s in an interval around a. Hence at time $t = 0$ the string is not disturbed around $x = a$. Let D be the distance from a to the nearest point on the string that is disturbed at time $t = 0$ (see figure). The previous discussion implies that the first time a feels the disturbance and is displaced from equilibrium is at time $t_0 = D/c$.

6.2. **The string with finite length.** The above ideas can be used to find the explicit solution for the BVP associated with the vibration of a string of finite length. Consider the following problem where $L = 20$ units.

$$\begin{array}{ll} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0 < x < 20, \quad t > 0 \\ u(x,0) = F(x), \quad u_t(x,0) = 0 & 0 < x < 20 \\ u(0,t) = 0, \quad u(20,t) = 0 & t > 0. \end{array}$$

We take the initial position to be given by a bump function at middle of the string:

$$F(x) = \begin{cases} ((x-10)^2-1)^2 & \text{if } 9 \leq x \leq 11 \\ 0 & \text{elsewhere} \end{cases}$$

To take advantage of the above ideas, let us think that this real string is a portion

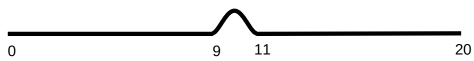

FIGURE 11. *Initial displacement of the string*

of an infinite (virtual) string. We are going to extend the function F which is defined over the interval $[0,\ 20]$ into a function defined over the whole real line $\mathbb{R}$. First we extend it as an odd function to the interval $[-20,\ 20]$ and then, we extend it to the whole real line as a periodic function. Let then F_{odd} to be the odd extension of F to $[-20,\ 20]$. Hence

$$\begin{array}{lll} F_{odd}(x) = F(x) & \text{for} & 0 \leq x \leq 20 \quad \text{and} \\ F_{odd}(x) = -F(-x) & \text{for} & -20 \leq x \leq 0. \end{array}$$

whose graph is given below. Now, we extend F_{odd} as a periodic function with period

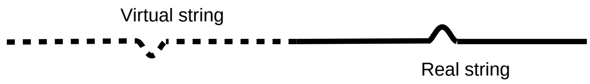

FIGURE 12. *Graph of F_{odd}*

40 to the whole real line as a function $\hat{F}$. The graph of $\hat{F}$ is given below Hence, F is

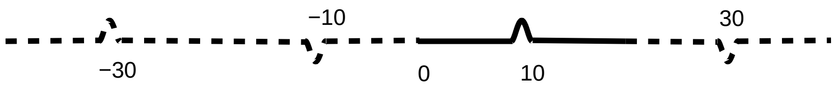

FIGURE 13. *Graph of $\hat{F}$*

defined on $[0,\ 20]$, F_{odd} on $[-20,\ 20]$, and $\hat{F}$ on $(-\infty\ ,\ \infty)$. The following relations hold

$$\hat{F}(s) = F(s) \qquad \forall s \in [0,\ 20]$$

$$\hat{F}(s) = -\hat{F}(-s) \quad \text{and} \quad \hat{F}(s \pm 40) = \hat{F}(s) \quad \forall s \in \mathbb{R}.$$

We know from the previous case that the function $\hat{u}(x,t)$ defined by

$$\hat{u}(x,t) = \frac{\hat{F}(x-ct) + \hat{F}(x+ct)}{2}$$

solves the initial value problem

$$\begin{array}{ll} \hat{u}_{tt}(x,t) = c^2 \hat{u}_{xx}(x,t) & \forall x \in \mathbb{R} \quad t > 0, \\ \hat{u}(x,0) = \hat{F}(x) \quad \text{and} \quad \hat{u}_t(x,0) = 0 & \forall x \in \mathbb{R} \end{array} .$$

Now consider only the x's that are on the real string $0 \le x \le 20$, and define u to be just the restriction of $\hat{u}$ to those x's. That is,

$$u(x,t) = \hat{u}(x,t) = \frac{\hat{F}(x-ct) + \hat{F}(x+ct)}{2} \quad x \in [0,\ 20] \quad t > 0.$$

We claim that in addition to the PDE and the two initial condition, this function u satisfies also the two endpoints conditions. Indeed, at the left end we have

$$u(0,t) = \frac{\hat{F}(-ct) + \hat{F}(ct)}{2} = 0 \quad (\text{because } \hat{F} \text{ is odd}).$$

To verify the condition at the right endpoint, we use the periodicity of $\hat{F}$ and its oddicity (?).

$$\hat{F}(20-ct) = \hat{F}(20-ct-40) = \hat{F}(-20-ct) = -\hat{F}(20+ct)$$

Hence

$$u(20,t) = \frac{\hat{F}(20-ct) + \hat{F}(20+ct)}{2} = 0.$$

The following figure illustrates the shape of the string at various times and in particular the periodicity of its vibrations. The period of vibration is

$$T = \frac{2 \times \text{Length of string}}{\text{Wave's speed}} = \frac{2L}{c}.$$

In our case, the period is $40/c$.

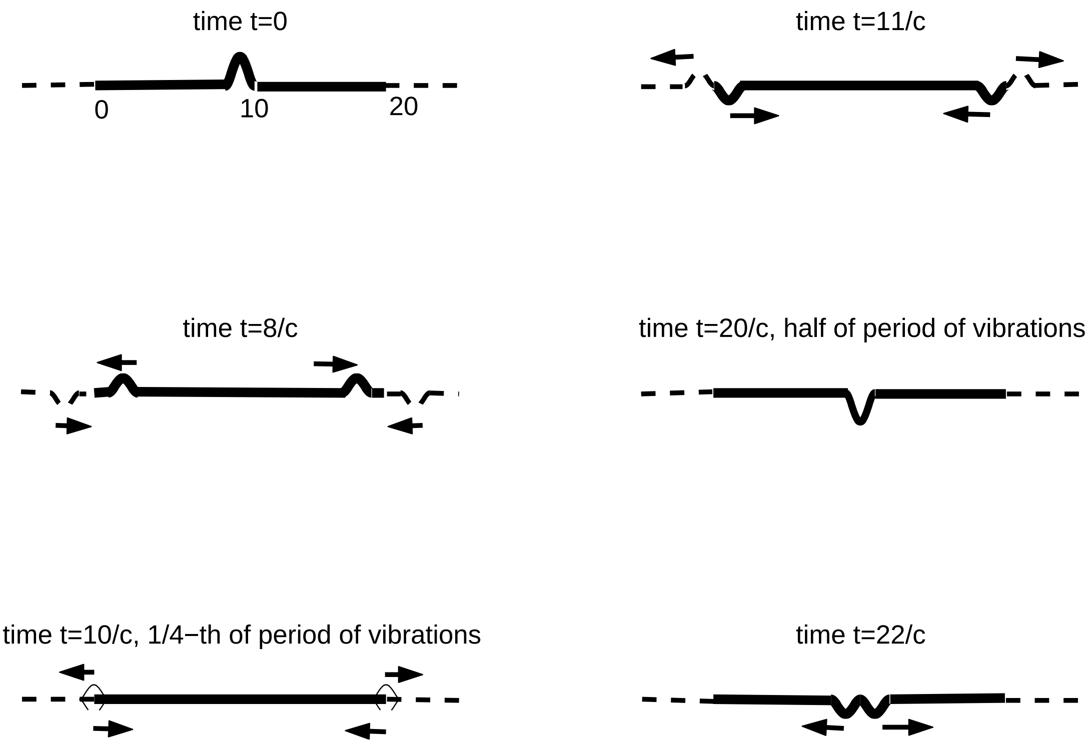

FIGURE 14. *Snapshots of the vibrating string*

Remark Note how the traveling waves, one to the left the other to the right cancel the effect of each other at the end points $x = 0$ and $x = 20$

7. Exercises

In exercises 1 to 6, write a BVP for the small vertical vibrations of a homogeneous string. Assume that the wave's speed is c, the length of the string is L and satisfies the following conditions:

Exercise 1. Both ends of the string are fixed on the x-axis, the initial position of the string is given by $f(x) = \sin(5\pi x/L)$ and is released from rest (plucked string). Take $L = 20$ and $c = .5$.

Exercise 2. Same characteristics as in exercise 1 but this time while the string is siting horizontally at equilibrium, it is struck (at time 0) with an initial velocity given by $g(x) = x(x-1)\sin(\pi x/L)$ (struck string).

Exercise 3. Take $L = \pi$ and $c = 2$. Both ends of the string are fixed and initially the string has position given by $f(x) = \sin x \cos 2x$ and velocity given by $g(x) = -1$.

Exercise 4. Take $L = 2\pi$, $c = 2$. Suppose that the right end is fixed while the left end is allowed to move, vertically, in such a way that its vertical displacement at time t is $0.2\sin(t)$. The string starts its motion from rest at equilibrium position.

Exercise 5. Same string as in exercise 4. This time suppose that the left end is fixed while the right end is allowed to move vertically. At the right end, the displacement at any time t is equal to the slope of the tangent line. Suppose that the string is set into motion from equilibrium position by a constant velocity $g(x) = 1$.

Exercise 6. Same string as in exercise 4. This time suppose that the right end is fixed while the left end is allowed to move vertically. At the left end, the tangent line at any time t is horizontal. Suppose that the string is set into motion from rest with an initial position given by the function $f(x) = \sin 3x$.

Exercises 7 to 10 deal with the small vertical vibrations of a homogeneous membrane. You are asked to write the corresponding BVP.

Exercise 7. The membrane is a square with side π. The boundary is attached in the (x, y)-plane, the wave's speed is $c = 1$. Initially, the membrane position is given by $f(x, y) = \mathrm{e}^{-y} \sin x \sin 2y$. The membrane is released from rest.

Exercise 8. The membrane is a rectangle with boundary attached to the (x, y)-plane. At equilibrium position, the membrane occupies the rectangle $[-5,\ 5] \times [-3,\ 3]$. Suppose that the membrane is set into motion by striking its center square $[-1,\ 1]^2$ by a constant velocity of magnitude 1 (so at time $t = 0$, each point inside the small square has velocity 1 while the other points of the membrane outside the small square have velocity 0). Take $c = .5$

Exercise 9. The membrane is a circular disk with radius 10 and with boundary fixed on the (x, y)-plane (take $c = 2$ here). The initial velocity of the membrane is 0 and its initial position is given in polar coordinates by the (bump) function

$$f(r, \theta) = \begin{cases} \mathrm{e}^{-r}(0.1 - r) & \text{if } 0 \le r \le 0.1 \\ 0 & \text{if } 0.1 < r \le 10. \end{cases}$$

Exercise 10. The membrane is a circular ring with inner radius 1 and outer radius 2 and $c = 1$. Suppose that the outer radius is fixed in the (x, y)-plane but the inner

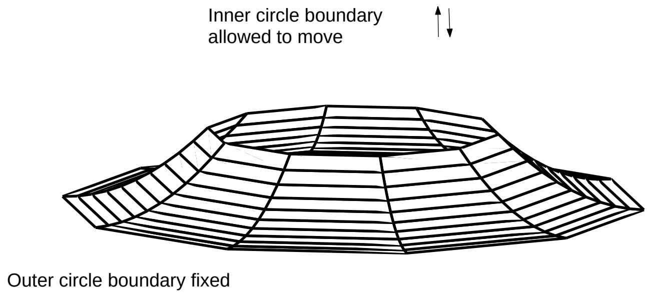

FIGURE 15. *Membrane in the shape of a ring*

radius is allowed to move vertically in such a way that at time t, each point of the inner radius has a displacement given by the function $\mathrm{e}^{-0.001t}\sin t$. The initial position and velocity are 0 (see figure).

Exercise 11. The vertical displacements of a string were found to be given by the function $u(x, t) = \sin 3t \cos(x/2)$. What is the corresponding wave's speed c?

Exercise 12. The vertical displacements of a membrane were found to be given by the function

$$u(x, y, t) = \cos(15t)\sin(3x)\cos(4y).$$

What is the corresponding wave's speed c?

Exercise 13. A string of length $L = 10$ and with fixed end on the horizontal axis is set into vertical motion by displacing it from its equilibrium position and then released from rest. The initial displacement is given by the function

$$f(x) = \begin{cases} 1 & \text{if } 4 < x < 6; \\ 0 & \text{if } 0 \le x \le 4 \text{ or } 6 \le x \le 10. \end{cases}$$

If $c = \sqrt{2}$, find the period of oscillations of the string. Use D'Alembert's method to find (and graph) the shape of the string at the following times $t = 1,\ 3,\ 5,\ 7,\ 9,\ 10$.

Exercise 14. (D'Alembert's method for the struck string). Consider the BVP

$$\begin{array}{lll} u_{tt} = c^2 u_{xx} & & 0 < x < L, \quad t > 0, \\ u(0, t) = u(L, t) = 0 & & t > 0, \\ u(x, 0) = 0 & u_t(x, 0) = g(x) & 0 < x < L. \end{array}$$

We are going to construct a solution for this BVP. First, let g_{odd} be the odd extension of the function g to the interval $[-L,\ L]$. Second, let $\hat{g}$ be the periodic extension of g_{odd} to $\mathbb{R}$. Hence, $\hat{g}$ has period $2L$. Third, let G be an antiderivative of $\hat{g}$. That is, $G'(s) = \hat{g}(s)$, for every $s \in \mathbb{R}$ and in particular, $G'(s) = g(s)$ if $0 \le s \le L$.

Verify that the function

$$u(x, t) = \frac{G(x + ct) - G(x - ct)}{2c}$$

solves the BVP.

Exercise 15. Use the D'Alembert's method of exercise 15 to illustrate the shape of the struck string if

$$g(x) = \cos x \quad L = \pi/2, \quad c = .5$$

Find (graph) the shape of the string at the following times $t = 0,\ \pi/4,\ \pi/2,\ \pi,\ 2\pi$.