THE LAPLACE EQUATION

The *Laplace* (or *potential*) *equation* is the equation

$$\Delta u = 0.$$

where Δ is the Laplace operator

$$\begin{aligned}
\Delta &= \frac{\partial^2}{\partial x^2} && \text{in } \mathbb{R} \\
\Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} && \text{in } \mathbb{R}^2 \\
\Delta &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} && \text{in } \mathbb{R}^3
\end{aligned}$$

The solutions u of the Laplace equation are called *harmonic functions* and play an important role in many areas of mathematics. The Laplace operator is one of the most important operators in mathematical physics. It is associated with the gravitational and electrical fields. For instance, we know from Newton's law of universal gravitation that two points A and B with masses M_A and M_B attract each other with forces $\overrightarrow{F}_A$ and $\overrightarrow{F}_B$ as in figure, each force with magnitude

$$F = \frac{GM_AM_B}{AB^2}$$

where G is the universal gravitational constant and AB is the distance from A to

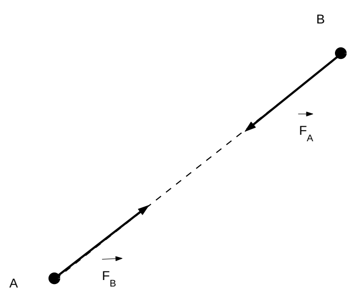

FIGURE 1. *Mass at A exert a force $\vec{F}_A$ on the mass at B*

B. As vectors these forces are opposite and we have

$$\overrightarrow{F_A} = F\frac{\overrightarrow{BA}}{AB} \quad \text{and} \quad \overrightarrow{F_B} = F\frac{\overrightarrow{AB}}{AB}$$

Suppose that A is located at the origin of the (x, y, z)-space and that B has a unit mass ($M_B = 1$) and it is located at the point (x, y, z), then the force $\overrightarrow{F}(x, y, z)$ with which A will attract B is

$$\overrightarrow{F} = C\left(\frac{x}{\sqrt{x^2+y^2+z^2}^3}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}^3}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}^3}\vec{k}\right)$$

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where the constant C is $C = -GM_A$. This vector-valued function $\overrightarrow{F}$ is the gravitational field generated by the point mass A.

The function Φ (which is real valued) and defined by

$$\Phi(x,y,z) = \frac{-C}{\sqrt{x^2+y^2+z^2}} = C(x^2+y^2+z^2)^{-1/2}$$

satisfies

$$\frac{\partial \Phi}{\partial x} = Cx(x^2+y^2+z^2)^{-3/2} = \frac{Cx}{\sqrt{x^2+y^2+z^2}^3}$$
$$\frac{\partial \Phi}{\partial y} = Cy(x^2+y^2+z^2)^{-3/2} = \frac{Cy}{\sqrt{x^2+y^2+z^2}^3}$$
$$\frac{\partial \Phi}{\partial z} = Cz(x^2+y^2+z^2)^{-3/2} = \frac{Cz}{\sqrt{x^2+y^2+z^2}^3}$$

Hence,

$$\overrightarrow{F}(x,y,z) = \overrightarrow{\text{grad}\Phi}(x,y,z).$$

The function Φ is called a *potential* of the vector field $\overrightarrow{F}$. Now, we compute the second partial derivatives of Φ. We use the notation $r = \sqrt{x^2+y^2+z^2}$. We have

$$\frac{\partial^2 \Phi}{\partial x^2} = C(r^{-3} - 3x^2 r^{-5})$$
$$\frac{\partial^2 \Phi}{\partial y^2} = C(r^{-3} - 3y^2 r^{-5})$$
$$\frac{\partial^2 \Phi}{\partial z^2} = C(r^{-3} - 3z^2 r^{-5})$$

When we add the three partial derivatives, we obtain

$$\Delta\Phi = C(3r^{-3} - 3(x^2+y^2+z^2)r^{-5}) = 0 \quad \text{since} \quad r^2 = x^2+y^2+z^2 \ .$$

This means that the potential Φ satisfies the Laplace equation. This is the reason why the Laplace equation is also referred to as the *potential equation*.

If we have N point masses $A_1 \cdots A_N$, each generates a gravitational field $\overrightarrow{F_i}$, $(i = 1, \cdots N)$ and each field has a potential Φ_i, the resulting field of all the mass points is the sum of the fields and the potential is the sum of the potentials[1] :

$$\sum_{i=1}^{N} \overrightarrow{F_i} = \sum_{i=1}^{N} \overrightarrow{\text{grad}\Phi_i}.$$

1. Some BVP For The Laplace Equation

The following are typical problems associated with the Laplace operator.

1.1. The Dirichlet Problem.

The problem is to find a harmonic function u inside a domain D so that the values of u are prescribed on the boundary ∂D of D ($u = f$ is given on the boundary ∂D).

[1]Suppose that a body occupies a region R in the (x,y,z)-space and has mass density $\rho(x,y,z)$ at the point (x,y,z). Then the body generates a gravitational field $\overrightarrow{F}$ whose potential function Φ is given by the following integral

$$\Phi(x,y,z) = \iiint_R \frac{\rho(\xi,\eta,\zeta)d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

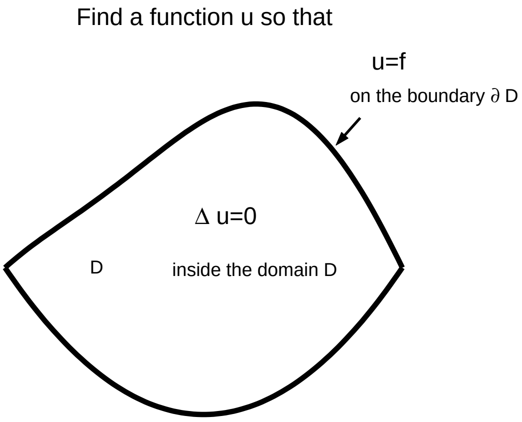

FIGURE 2. *The Dirichlet Problem*

1.2. **The Neumann Problem.** The problem is to find a harmonic function u inside the domain D so that the normal derivatives of u, (i.e. $\frac{\partial u}{\partial \eta}$) are prescribed on the boundary ($\frac{\partial u}{\partial \eta} = g$ on ∂D.) Recall that the normal derivative at a point

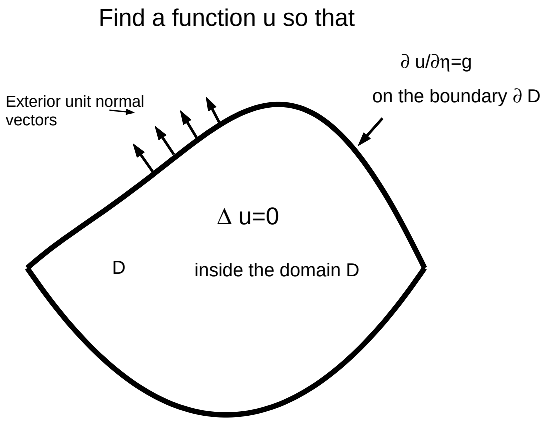

FIGURE 3. *The Neumann Problem*

(x, y) on the boundary ∂D is

$$\frac{\partial u}{\partial \eta}(x, y) = \overrightarrow{\mathrm{grad} u}(x, y) \cdot \overrightarrow{n}(x, y)$$

where $\overrightarrow{n}(x, y)$ is the exterior unit normal at the point (x, y).

1.3. **The Problem with mixed boundary conditions.** The problem is to find a harmonic function u inside the domain D so that on the boundary ∂D it satisfies $au + b\frac{\partial u}{\partial \eta} = h$, where a, b, and h are given.

2. Steady-State Temperature Problems

The above problems for the Laplace equation are illustrated by the steady-state solutions of the 2-D and 3-D heat equation. By a *steady-state* function u, we mean a function that is independent on time t. Thus, $u_t \equiv 0$. In particular if u satisfies the heat equation $u_t = \Delta u$ and u is steady-state, then it satisfies

$$\Delta u = 0.$$

2.1. Example 1.

Example 1. Write the BVP for the steady-state temperature $u(x, y)$ in a 1×2 rectangular plate if the bottom horizontal side is kept at 0^0, the top horizontal side at 100^0, the left vertical side at -10^0 and the right vertical side at 200^0.

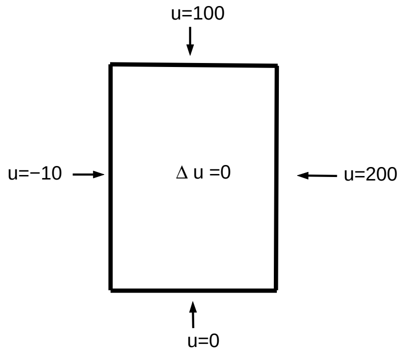

Figure 4. *A Dirichlet problem for the steady-state temperature*

This is an example of a Dirichlet problem. We can write it as

$$\begin{array}{lll} \Delta u(x,y) = 0, & & 0 < x < 1, \quad 0 < y < 2; \\ u(x,0) = 0, & u(x,2) = 100, & 0 < x < 1; \\ u(0,y) = -10, & u(1,y) = 200, & 0 < y < 2. \end{array}$$

2.2. Example 2.

Example 2. This time we have steady-state temperature in a 1×2 rectangular plate. Assume that the boundary conditions are as follows: the bottom and right sides are insulated and left and top sides are kept at constant temperatures of 0 and 100 degrees, respectively.

The BVP can be written as

$$\begin{array}{lll} \Delta u(x,y) = 0, & & 0 < x < 1, \quad 0 < y < 2; \\ \dfrac{\partial u}{\partial y}(x,0) = 0, & u(x,2) = 100, & 0 < x < 1; \\ u(0,y) = 0, & \dfrac{\partial u}{\partial x}(1,y) = 200, & 0 < y < 2. \end{array}$$

2.3. Example 3.

Example 3. Consider a plate in the shape of a quarter of a circle with radius 1. Suppose that the temperature is steady-state, the circular side is insulated, one radial side is kept at 100 degrees and the other at 50 degrees.

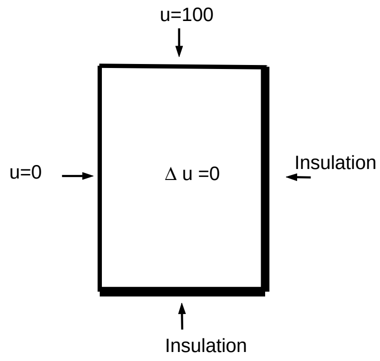

FIGURE 5. *A mixed problem for the steady-state temperature*

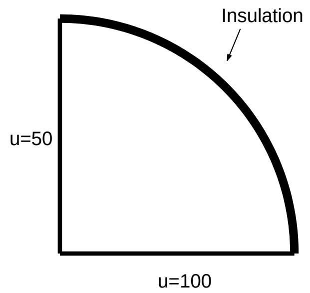

FIGURE 6. *A BVP for the steady-state temperature in a circular domain*

The BVP can be written as

$$\begin{array}{ll} \Delta u(x,y) = 0, & 0 < x^2 + y^2 < 1;\ x > 0,\ y > 0 \\ u(x,0) = 50, & 0 < x < 1; \\ u(0,y) = 100, & 0 < y < 1; \\ \dfrac{\partial u}{\partial \eta}(x,y) = 0, & x^2 + y^2 = 1,\ x > 0,\ y > 0. \end{array}$$

We express the normal derivative $\dfrac{\partial u}{\partial \eta}$ in terms of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ by using

$$\frac{\partial u}{\partial \eta} = \overrightarrow{\text{grad} u} \cdot \vec{n}\,,$$

where $\vec{n}$ is the unit normal vector to the unit circle. For a circle $x^2 + y^2 = R^2$ the unit normal at point (x_0, y_0) is

$$\vec{n}(x_0, y_0) = \frac{x_0}{R}\vec{i} + \frac{y_0}{R}\vec{j}.$$

In our case $R = 1$, so that at each point (x, y) on the boundary $(x^2 + y^2 = 1)$ the unit normal vector is just.

$$\vec{n}(x,y) = x\vec{i} + y\vec{j}.$$

Hence,

$$\frac{\partial u}{\partial \eta}(x,y) = x\frac{\partial u}{\partial x}(x,y) + y\frac{\partial u}{\partial y}(x,y)$$

. The BVP can then be rewritten as

$$\begin{array}{ll}\Delta u(x,y)=0, & 0<x^2+y^2<1;\ x>0,\ y>0\\ u(x,0)=50, & 0<x<1;\\ u(0,y)=100, & 0<y<1;\\ x\dfrac{\partial u}{\partial x}(x,y)+y\dfrac{\partial u}{\partial x}(x,y)=0, & x^2+y^2=1,\ x>0,\ y>0.\end{array}$$

Remark Since the domain is circular, this problem is in fact better suited for polar coordinates. We will revisit these problems.

3. The Laplacian In Polar, Cylindrical, And Spherical Coordinates

For BVP that deal with non rectangular shaped domains, it is useful to use coordinates systems other than the rectangular coordinates. In particular, for cylindrically or spherically shaped domains, the appropriate coordinates are the cylindrical and spherical coordinates. To use these coordinates, it is necessary to express the Laplace operator Δ in these coordinates.

3.1. **The 2D-Laplacian in polar coordinates.** First recall that a point $p \in \mathbb{R}^2$ can be expressed in rectangular coordinates as (x, y) or in polar coordinates as (r, θ)

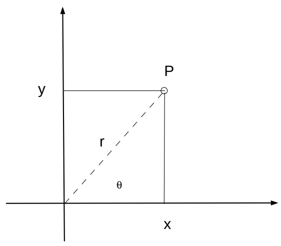

Figure 7. *Rectangular and polar coordinates*

The relations between these coordinates is given by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

and

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \text{ or } \cot\theta = \frac{x}{y}$$

Let u be a function defined in the the plane $\mathbb{R}^2$. Then u can be expressed in terms of the rectangular coordinates as $u(x, y)$ or in terms of the polar coordinates $u(r, \theta)$. Its Laplacian Δu is also a function in $\mathbb{R}^2$. We know how to express it in rectangular coordinates:

$$\Delta u(x, y) = u_{xx}(x, y) + u_{yy}(x, y).$$

We would like to express Δu in polar coordinates only (so that x and y will not appear at all but only r and θ are involved). For this we need to use the chain rule to relate the first and second partial derivatives of u given in terms of x, y to their counterparts in terms of r and θ.

By differentiating r^2 with respect to x and y, we obtain

$$2rr_x = 2x \quad \Rightarrow \quad r_x = \frac{x}{r} = \cos\theta$$
$$2rr_y = 2y \quad \Rightarrow \quad r_y = \frac{y}{r} = \sin\theta$$

Then

$$r_{xx} = \frac{r - xr_x}{r^2} = \frac{r - x\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} = \frac{\sin^2\theta}{r}$$
$$r_{yy} = \frac{r - yr_y}{r^2} = \frac{r - y\frac{y}{r}}{r^2} = \frac{r^2 - y^2}{r^3} = \frac{x^2}{r^3} = \frac{\cos^2\theta}{r}$$

Now we differentiate $\tan\theta$ with respect to x and y

$$(\tan\theta)_x = (\sec^2\theta)\,\theta_x = -\frac{y}{x^2} \qquad \Rightarrow \qquad \theta_x = -\frac{y\cos^2\theta}{x^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$
$$(\tan\theta)_y = (\sec^2\theta)\,\theta_y = \frac{1}{x} \qquad \Rightarrow \qquad \theta_y = \frac{\cos^2\theta}{x} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

Then for the second partial derivatives, we get

$$\theta_{xx} = \left(-\frac{y}{r^2}\right)_x = \frac{2yrr_x}{r^4} = \frac{2y\frac{x}{r}}{r^3} = \frac{2xy}{r^4} = \frac{2\sin\theta\cos\theta}{r^2}$$
$$\theta_{yy} = \left(\frac{x}{r^2}\right)_y = -\frac{2xrr_y}{r^4} = -\frac{2x\frac{y}{r}}{r^3} = -\frac{2xy}{r^4 3} = -\frac{2\sin\theta\cos\theta}{r^2}$$

Now we go back to a function u defined in the plane and relate its derivatives from one system of coordinates to the other by using the chain rule. We have

$$u_x = u_r r_x + u_\theta \theta_x \quad \text{and} \quad u_y = u_r r_y + u_\theta \theta_y.$$

For the second derivatives, we have

$$\begin{aligned} u_{xx} &= (u_x)_x = (u_r r_x + u_\theta\theta_x)_x = (u_r)_x r_x + (u_\theta)_x\theta_x + u_r r_{xx} + u_\theta\theta_{xx} \\ &= [(u_r)_r r_x + (u_r)_\theta\theta_x]r_x + [(u_\theta)_r r_x + (u_\theta)_\theta\theta_x]\theta_x + u_r r_{xx} + u_\theta\theta_{xx} \end{aligned}$$

Hence,

$$u_{xx} = u_{rr}(r_x)^2 + 2u_{r\theta}r_x\theta_x + u_{\theta\theta}(\theta_x)^2 + u_r r_{xx} + u_\theta\theta_{xx}$$

Similarly, we have

$$u_{yy} = u_{rr}(r_y)^2 + 2u_{r\theta}r_y\theta_y + u_{\theta\theta}(\theta_y)^2 + u_r r_{yy} + u_\theta\theta_{yy}.$$

By adding these last relations, we obtain

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} \\ &= u_{rr}((r_x)^2 + (r_y)^2) + 2u_{r\theta}(r_x\theta_x + r_y\theta_y) \\ &\quad + u_{\theta\theta}((\theta_x)^2 + (\theta_y)^2) + u_r(r_{xx} + r_{yy}) + u_\theta(\theta_{xx} + \theta_{yy}) \end{aligned}$$

By using the formulas given above for the derivatives of r and θ, we get

$$(r_x)^2 + (r_y)^2 = 1, \quad r_x\theta_x + r_y\theta_y = 0, \; r_{xx} + r_{yy} = \frac{1}{r},$$

$$(\theta_x)^2 + (\theta_y)^2 = \frac{1}{r^2}, \quad \text{and} \quad \theta_{xx} + \theta_{yy} = 0.$$

With these, the expression for Δu becomes

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

The right expression contains only the variables r and θ. We have established the following

Proposition 3.1. *The Laplace operator in polar coordinates is:*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}.$$

Example Consider a plate in the shape of a 45^0-sector of a ring with inner radius 1 and outer radius 2. Suppose that the steady state temperature in the plate satisfies the boundary conditions as shown in the figure. To write the BVP for the steady

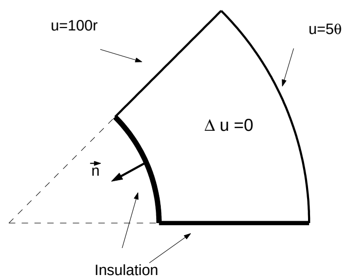

FIGURE 8. *Steady-state temperature in a* 45^0 *sector*

state temperature, we need to

- write the PDE inside the sector (Laplace equation)

$$u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, \qquad 1 < r < 2, \quad 0 < \theta < \pi/4.$$

- write the specified temperature on the slanted edge

$$u(r,\pi/4) = 100r \qquad 1 < r < 2.$$

- write the specified temperature on the outer circular side

$$u(2,\theta) = 5\theta, \qquad 0 < \theta < \pi/4.$$

- write the insulation condition on the horizontal edge
- write the insulation condition on the inner circular side

The last two condition need an explanation. Recall that insulating a surface means that the normal derivative of the temperature u is 0. Now for the horizontal side, it means that $u_y(x,0) = 0$. But, we need to write this in polar coordinates. At each point $(x,0)$, we have then

$$0 = u_y = u_r r_y + u_\theta \theta_y$$

From the previous calculation we have $r_y = y/r$ and so $r_y = 0$ when $y = 0$ (and of course $r > 0$). Also, $\theta_y = x/r^2 \neq 0$ since $x > 1$. Hence, the insulation condition on the horizontal side is simply $u_\theta(r,0) = 0$ for $1 < r < 2$.

The last condition is $\dfrac{\partial u}{\partial \eta} = 0$ on the inner circular side. Here, $\frac{\partial u}{\partial \eta}$ is the normal derivative. The outer unit normal to the inner circle is simply $\vec{n} = -(1/r)(x\vec{i} + y\vec{j})$. Hence,

$$\begin{aligned}\frac{\partial u}{\partial \eta} &= \overrightarrow{\text{grad}} u \cdot \vec{n} = -\frac{1}{r}(xu_x + yu_y)\\ &= -\frac{1}{r}\left(x(u_r r_x + u_\theta \theta_x) + y(u_r r_y + u_\theta \theta_y)\right)\\ &= -\frac{xr_x + yr_y}{r} u_r - \frac{x\theta_x + y\theta_y}{r} u_\theta\end{aligned}$$

From the above calculation, we have

$$\frac{xr_x + yr_y}{r^2} = 1, \quad \text{and} \quad x\theta_x + y\theta_y = 0.$$

All of this simply means that in polar coordinates $\dfrac{\partial u}{\partial \eta} = u_r$. Therefore, the insulation of the inner circle reads

$$u_r(1, \theta) = 0, \qquad 0 < \theta < \pi/4.$$

Now, we can write the BVP as

$$\begin{array}{ll} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 1 < r < 2, \;\; 0 < \theta < \pi/4;\\ u(r, \pi/4) = 100r, & 1 < r < 2;\\ u(2,\theta) = 5\theta, & 0 < \theta < \pi/4;\\ u_\theta(r, 0) = 0, & 1 < r < 2;\\ u_r(1,\theta) = 0, & 0 < \theta < \pi/4. \end{array}$$

3.2. **The 3-D Laplacian in cylindrical coordinates.** Recall that if a point p in $\mathbb{R}^3$ has cartesian coordinates (x, y, z), then its cylindrical coordinates are (r, θ, z) with r and θ as above:

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

From the previous calculations, we get the following

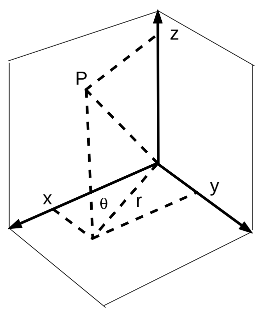

FIGURE 9. *Rectangular and cylindrical coordinates*

Proposition 3.2. *The expression for the three dimensional Laplacian in cylindrical coordinates is*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

3.3. **The 3-D Laplacian in spherical coordinates.** Recall that if a point p in $\mathbb{R}^3$ has cartesian coordinates (x, y, z), then its spherical coordinates are (ρ, θ, ϕ) (see figure) with

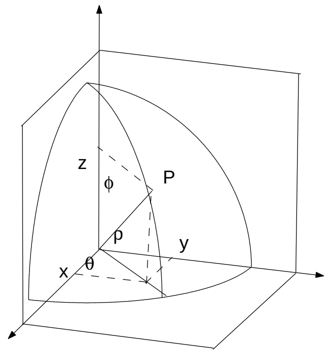

FIGURE 10. *Rectangular and spherical coordinates*

$$x = \rho \cos\theta \sin\phi, \qquad y = \rho \sin\theta \sin\phi, \quad z = \rho \cos\phi.$$

We would like to express the Laplacian Δ in terms of only ρ, θ, and ϕ. This can be achieved by using the chain rule and so we would need to compute the first and second partial derivatives ρ, θ, ϕ with respect to x, y, z. This is not difficult to do, but there is a more economical (just a bit more economical) way to reach the same result by using the transition from cylindrical (r, θ, z) to spherical coordinates (ρ, θ, ϕ) instead of going from rectangular to spherical. The transition from these systems of coordinates is given by

$$r = \rho \sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi$$

and

$$\rho^2 = r^2 + z^2, \quad \theta = \theta, \quad \tan\phi = \frac{r}{z}.$$

Notice that the coordinate θ is the same in both systems and so $\theta_r = \theta_\phi = 0$. This is why it is a little bit easier to use this transition.

From the previous section, we know the expression of Δ in cylindrical coordinates. The action of Δ on a function u is:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + u_{zz} + \frac{1}{r^2}u_{\theta\theta}.$$

We need therefore to express u_r, u_{rr}, and u_{zz} in terms of the spherical variables. By the chain rule, we have the following

$$\begin{aligned}
u_r &= u_\rho \rho_r + u_\phi \phi_r \qquad \text{(no } \theta \text{ involved!);} \\
u_z &= u_\rho \rho_z + u_\phi \phi_z; \\
u_{rr} &= (u_\rho)_r \rho_r + (u_\phi)_r \phi_r + u_\rho \rho_{rr} + u_\phi \phi_{rr} \\
&= \left(u_{\rho\rho}\rho_r + u_{\rho\phi}\phi_r\right)\rho_r + \left(u_{\phi\rho}\rho_r + u_{\phi\phi}\phi_r\right)\phi_r + u_\rho\rho_{rr} + u_\phi\phi_{rr} \quad \text{so} \\
u_{rr} &= u_{\rho\rho}(\rho_r)^2 + 2u_{\rho\phi}\rho_r\phi_r + u_{\phi\phi}(\phi_r)^2 + u_\rho\rho_{rr} + u_\phi\phi_{rr} \\
u_{zz} &= u_{\rho\rho}(\rho_z)^2 + 2u_{\rho\phi}\rho_z\phi_z + u_{\phi\phi}(\phi_z)^2 + u_\rho\rho_{zz} + u_\phi\phi_{zz}
\end{aligned}$$

From these, we get then

$$\begin{aligned}
u_{rr} + \tfrac{1}{r}u_r + u_{zz} &= u_{\rho\rho}\left({\rho_r}^2 + {\rho_z}^2\right) + 2u_{\rho\phi}\left(\rho_r\phi_r + \rho_z\phi_z\right) + u_{\phi\phi}\left({\phi_r}^2 + {\phi_z}^2\right) \\
&\quad + u_\rho\left(\rho_{rr} + \rho_{zz} + \tfrac{1}{r}\rho_r\right) + u_\phi\left(\phi_{rr} + \phi_{zz} + \tfrac{1}{r}\phi_r\right)
\end{aligned}$$

Now we express the terms between parentheses in terms of spherical coordinates. We have,

$$\begin{array}{llll} \rho_r = \sin\phi, & \rho_z = \cos\phi, & \rho_{rr} = \dfrac{\cos^2\phi}{\rho}, & \rho_{zz} = \dfrac{\sin^2\phi}{\rho}, \\ \phi_r = \dfrac{\cos\phi}{\rho}, & \phi_z = -\sin\phi\rho, & \phi_{rr} = -2\dfrac{\sin\phi\cos\phi}{\rho^2}, & \phi_{zz} = 2\dfrac{\sin\phi\cos\phi}{\rho^2} \end{array}$$

We obtain the parentheses terms as

$$\rho_r{}^2 + \rho_z{}^2 = 1, \quad \rho_r\phi_r + \rho_z\phi_z = 0, \quad \phi_r{}^2 + \phi_z{}^2 = \frac{1}{\rho^2},$$
$$\rho_{rr} + \rho_{zz} + \frac{1}{r}\rho_r = \frac{2}{\rho}, \quad \text{and} \quad \phi_{rr} + \phi_{zz} + \frac{1}{r}\phi_r = \frac{\cot\phi}{\rho^2}$$

This gives then

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot\phi}{\rho^2}u_\phi.$$

To obtain Δu, need to add the term $\frac{1}{r^2}u_{\theta\theta}$. This gives the following

Proposition 3.3. *The expression of the Laplacian in spherical coordinates is*

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial}{\partial\phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}$$

Example Consider the steady-state temperature in the portion of a spherical shell contained in the first octant. Assume that that the inner radius is 1 and the outer radius is 2 and that the boundary conditions are as indicated in the figure. The

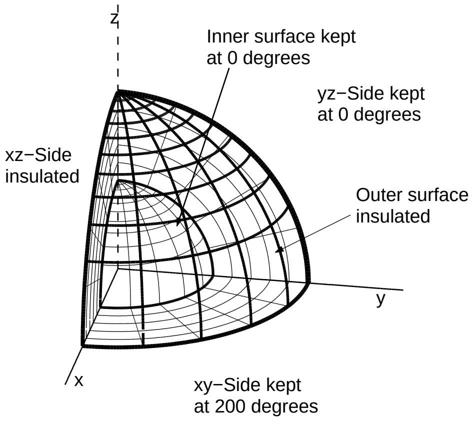

FIGURE 11. *Portion of a spherical shell*

steady-state temperature $u(\rho,\theta,\phi)$ satisfies

- The Laplace equation inside the solid
$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot\phi}{\rho^2}u_\phi + \frac{1}{\rho^2\sin^\phi}u_{\theta\theta} = 0$$
- The xy-face: $u = 200$
- The yz-face: $u = 0$

- The xz-face: $u_\theta = 0$
- The inner spherical boundary $u = 0$
- The outer spherical boundary $u_\rho = 0$

The BVP is therefore

$$\begin{array}{ll}
u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot\phi}{\rho^2}u_\phi + \frac{1}{\rho^2 \sin^\phi}u_{\theta\theta} = 0 & 1 < \rho < 2,\ 0 < \theta < \pi/2,\ 0 < \phi < \pi/2; \\
u(\rho, \theta, \pi/2) = 200 & 1 < \rho < 2,\ 0 < \theta < \pi/2; \\
u(\rho, \pi/2, \phi) = 0 & 1 < \rho < 2,\ 0 < \phi < \pi/2; \\
u_\theta(\rho, 0, \phi) = 0 & 1 < \rho < 2,\ 0 < \phi < \pi/2; \\
u(1, \theta, \phi) = 0 & 0 < \theta < \pi/2,\ 0 < \phi < \pi/2; \\
u_\rho(2, \theta, \phi) = 0 & 0 < \theta < \pi/2,\ 0 < \phi < \pi/2.
\end{array}$$

4. Exercises

All of the following exercises deal with the steady-state distribution of the temperature in either 2-dimensional plates or 3-dimensional regions. For each exercise, write the corresponding BVP.

Exercise 1. A 10×20 rectangular plate. The vertical sides are kept at constant temperatures. The left at 10 degrees and the right at 50 degrees. The horizontal sides are insulated. (Can you guess the solution for this problem?)

Exercise 2. A 10×20 rectangular plate with boundary conditions as in figure. At the lower side where there is poor insulation the normal derivative of the temperature is equal to 0.5 times the temperature.

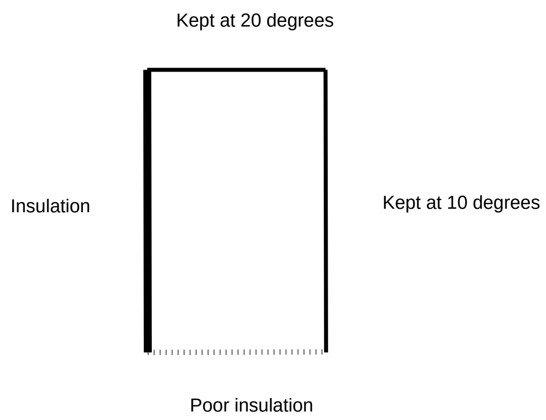

Figure 12. *Rectangular plate*

Exercise 3. A $1 \times 2 \times 3$ solid rectangular box. with boundary conditions as indicated in the figure.

Exercise 4. A plate in the shape of a quarter of a disk of radius 10 and with boundary conditions as indicated in the figure.

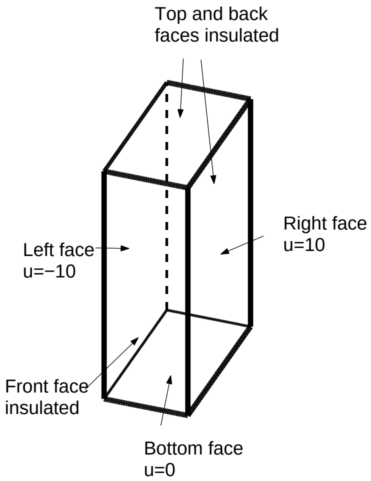

FIGURE 13. *Rectangular box*

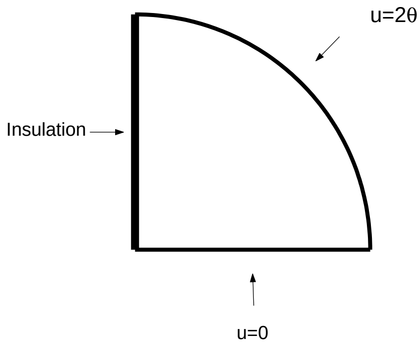

FIGURE 14. *A quarter of a disk shaped plate*

Exercise 5. A plate in the shape of half of a ring with inner radius 1 and outer radius b, (with $b > 1$), where the boundary conditions are as indicated in the figure.

Exercise 6. A plate in the shape of a 45^0-sector of a ring with radii 1 and b, with $b > 1$, and where the boundary conditions are as indicated

Exercise 7. A solid cylinder of radius 10 and height 20. The top surface is insulated, the bottom surface is kept at temperature 20 degrees and the lateral surface is kept at temperature 100 degrees.

Exercise 8. A solid hollow cylinder (cylindrical shell) with radii 10 and 15 and with height 20. Assume that the inner lateral surface is insulated, the outer lateral surface is kept at temperature 100 degrees, the bottom surface is insulated and the top surface is kept at 50 degrees.

Exercise 9. A solid sphere with radius 10. The top hemisphere is kept at temperature 100 degrees and the lower hemisphere is kept at temperature 0 degrees.

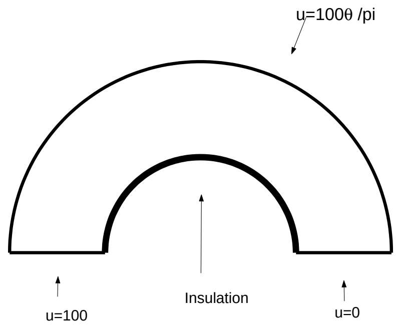

FIGURE 15. *A half-ring shaped plate*

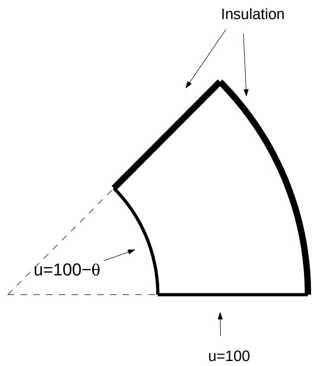

FIGURE 16. *A* 45^0 *sector of a ring shaped plate*

Exercise 10. A hollow solid sphere (spherical shell) with radii 1 and 5. The inner and outer surfaces are kept at temperatures 100 and 50 degrees, respectively. (Can you guess the temperature distribution?)