CLASSIFICATION AND PRINCIPLE OF SUPERPOSITION FOR SECOND ORDER LINEAR PDE

1. Linear Partial Differential Equations

A partial differential equation (PDE) is an equation, for an unknown function u, that involves independent variables, x, y, $\cdots$, the function u, and the partial derivatives of u. The *order* of the PDE is the order of the highest partial derivative of u in the equation.

The following are examples of some famous PDE's

$$\begin{array}{lll}
u_t - k(u_{xx} + u_{yy}) = 0 & \text{2d heat equation, order 2} & (1)\\
u_{tt} - c^2(u_{xx} + u_{yy}) = 0 & \text{2d wave equation, order 2} & (2)\\
u_{xx} + u_{yy} + u_{zz} = 0 & \text{3d Laplace equation, order 2} & (3)\\
u_{xx} + u_{yy} + u_{zz} + \lambda u = 0 & \text{Helmholtz equation, order 2} & (4)\\
u_{xx} + x u_{yy} = 0 & \text{Tricomi equation, order 2} & (5)\\
\mathrm{i}u_t + u_{xx} + u_{yy} + u_{zz} = 0 & \text{Schrödinger's equation, order 2} & (6)\\
u_{tt} + u_{xxxx} = 0 & \text{Beam quation, order 4} & (7)\\
u_x^2 + u_y^2 = 1 & \text{Eikonal equation, order 1} & (8)\\
u_t - u_{xx} + u u_x = 0 & \text{Burger's equation, order 2} & (9)\\
u_t - 6u u_{xx} + u_{xxx} = 0 & \text{KdV equation, order 3} & (10)
\end{array}$$

and the following are other examples (of non famous PDEs that I just made up)

$$\begin{array}{lll}
u_x + \sin(u_y) = 0 & \text{order 1} & (11)\\
3x^2 \sin(xy)\mathrm{e}^{-y^2} u_{xx} + \ln(x^2 + y^2) u_y = 0 & \text{order 2} & (12)
\end{array}$$

A PDE is said to be *linear* if it is linear in u and its partial derivatives (it is a first degree polynomial in u and its derivatives). In the above lists, equations (1) to (7) and (12) are linear PDES while equations (8) to (11) are nonlinear PDEs.

The general form of a first order linear PDE in two variables x, y is:

$$A(x, y)u_x + B(x, y)u_y + C(x, y)u = f(x, y)$$

and that of a first order linear PDE in three variables x, y, z is:

$$A(x, y, z)u_x + B(x, y, z)u_y + C(x, y, z)u_z + D(x, y, z)u = f(x, y, z)$$

The general form of a second order linear PDE in two variables is:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f, \qquad (13)$$

where the coefficients A, B, C, D, E, F and the right hand side f are functions of x and y. If the coefficients A,B,C, D, E, and F are constants, the equation is said to be *a linear PDE with constant coefficients.*

Date: January 26, 2016.

2. Linear Partial Differential Operator

The linear PDE (13) can be written in a more compact form as

$$Lu = f, \qquad (14)$$

where L is the *differential operator* defined by

$$\begin{aligned} L \quad &= A(x,y)\frac{\partial^2}{\partial x^2} + 2B(x,y)\frac{\partial^2}{\partial x \partial y} + C(x,y)\frac{\partial^2}{\partial y^2} + \\ &\qquad + D(x,y)\frac{\partial}{\partial x} + E(x,y)\frac{\partial}{\partial y} + F(x,y). \qquad (15) \end{aligned}$$

Remark. L operates on functions as a mapping $u \longrightarrow Lu$ from the space of functions into itself. It transforms a function u into another function given by $Au_{xx} + 2Bu_{xy} + \cdots$, whence the terminology.

The operator L is linear (as a transformation from the space of function into itself). More precisely, we have the following

Lemma. *For any two functions u and v (with second order partial derivatives) and for any constant c, we have*

$$L(u+v) = L(u) + L(v) \quad \text{and} \quad L(cu) = cLu.$$

Proof. By using the linearity of the differentiation ($(u+v)_x = u_x + v_x$, $(u+v)_{xx} = u_{xx} + u_{yy}$, etc.) and after grouping together the terms containing u, and grouping together the terms containing v, we get

$$\begin{aligned} L(u+v) \quad &= A(u+v)_{xx} + 2B(u+v)_{xy} + C(u+v)_{yy} + \\ &\qquad + D(u+v)_x + E(u+v)_y + F(u+v) \\ &= (Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu) + \\ &\qquad + (Av_{xx} + 2Bv_{xy} + Cv_{yy} + Dv_x + Ev_y + Fv) \\ &= Lu + Lv \end{aligned}$$

This verifies the first property. The same argument applies for the second property.

Remark. The linearity of L can be simply expressed as

$$L(au + bv) = aLu + bLv$$

for every pair of functions u, v and constants a, b.

The operators associated with the Laplace, wave, and heat equations are:

Laplace Operator

$$\begin{aligned} \Delta \quad &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} && \text{in } \mathbb{R}^2 \\ \Delta \quad &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} && \text{in } \mathbb{R}^3 \end{aligned}$$

Wave Operator (usually denoted $\Box$, and normalized with $c = 1$)

$$\begin{aligned}
\Box &= \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} && \text{in 1 space variable}\\
\Box &= \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial^2}{\partial t^2} - \Delta && \text{in 2 space variables}\\
\Box &= \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{\partial^2}{\partial t^2} - \Delta && \text{in 3 space variables}
\end{aligned}$$

Heat Operator (usually denoted H and normalized with $k = 1$)

$$\begin{aligned}
H &= \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} && \text{in 1 space variable}\\
H &= \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial}{\partial t} - \Delta && \text{in 2 space variables}\\
H &= \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{\partial}{\partial t} - \Delta && \text{in 3 space variables}
\end{aligned}$$

Remark. A linear PDE of order m in $\mathbb{R}^n$ is a PDE of the form

$$Lu(x) = f(x), \qquad x = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

where L is the m-th order linear differential operator given by

$$L = \sum_{k_1+\cdots+k_n \le m} a_{k_1,\cdots,k_n}(x) \frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}},$$

where the coefficients $a_{k_1,\cdots,k_n}$ are functions of x.

3. Classification

Consider in $\mathbb{R}^2$ a second order linear differential operator L as in (15). To the operator L, we associate the *discriminant* $\mathbb{D}(x, y)$ given by

$$\mathbb{D}(x, y) = A(x, y)C(x, y) - B(x, y)^2.$$

The operator L (or equivalently, the PDE $Lu = f$) is said to be:

- *elliptic* at the point (x_0, y_0), if $\mathbb{D}(x_0, y_0) > 0$;
- *hyperbolic* at the point (x_0, y_0), if $\mathbb{D}(x_0, y_0) < 0$;
- *parabolic* at the point (x_0, y_0), if $\mathbb{D}(x_0, y_0) = 0$.

If L is elliptic (resp. hyperbolic, parabolic) at each point (x, y) in a domain $\Omega \subset \mathbb{R}^2$, then L is said to be elliptic (resp. hyperbolic, parabolic) in Ω.

The 2-dimensional Laplace operator Δ is elliptic in $\mathbb{R}^2$ (we have $\mathbb{D} \equiv 1$). The 1-dimensional wave operator $\Box$ is hyperbolic in $\mathbb{R}^2$ (we have $\mathbb{D} \equiv -1$). The 2-dimensional heat operator H is parabolic in $\mathbb{R}^2$ (we have $\mathbb{D} \equiv 0$). These three operators Δ, $\Box$, and H are prototype operators. They are prototype in the following sense. If an operator L is elliptic in a region Ω, then the solutions of the equation $Lu = 0$ behave as those of the equation $\Delta u = 0$; if L is hyperbolic in a region Ω, then the solutions of the equation $Lu = 0$ behave as those of the equation $\Box u = 0$; and if L is parabolic in a region Ω, then the solutions of the equation $Lu = 0$ behave as those of the equation $Hu = 0$.

When the coefficients of an operator L are not constant, the type of L might vary from point to point. An example is given by the Tricomi operator

$$T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}.$$

The type of T is illustrated in the figure. The discriminant of T is $\mathbb{D} = x$. Hence, T

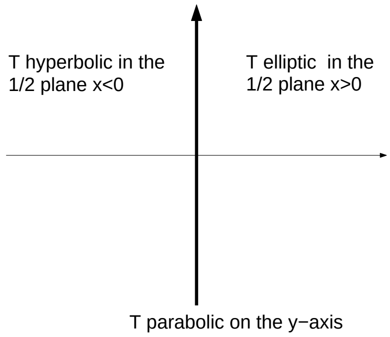

Figure 1. *Type of the Tricomi operator*

is elliptic in the half-plane $x > 0$, hyperbolic in the half-plane $x < 0$, and parabolic on the y-axis.

Remark about the terminology Consider again the operator L given in (15). We define its symbol at the point (x_0, y_0) as the polynomial $P(\xi, \eta)$ obtained from (15) by replacing $\partial/\partial x$ by the variable ξ and the by replacing $\partial/\partial y$ by the variable η. The result, after evaluating the coefficients $A, \cdots, F$ at the point (x_0, y_0), is the polynomial with 2 variables

$$P(\xi, \eta) = A\xi^2 + 2B\xi\eta + C\eta^2 + D\xi + E\eta + F.$$

Now if we consider the curves in the (ξ, η)-plane, given by the equation

$$P(\xi, \eta) = \text{ constant},$$

then these curves are ellipses if $\mathbb{D}(x_0, y_0) > 0$; hyperbolas if $\mathbb{D}(x_0, y_0) < 0$; and parabolas if $\mathbb{D}(x_0, y_0) = 0$. This justifies the terminology for the type of an operator.

Second order operators in $\mathbb{R}^3$: The classification for second order linear operators in $\mathbb{R}^3$ (or in higher dimensional spaces) is done in an analogous way by associating to the operator its symbol which is a polynomial of degree two in three variables and considering the surfaces defined by the level sets of the polynomial. These surfaces are either ellipsoids; hyperboloids; or paraboloids. The operator is accordingly labeled as elliptic, hyperbolic, or parabolic.

This is equivalent to the following. Consider a second order operator L in $\mathbb{R}^3$

$$L = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + 2c\frac{\partial^2}{\partial x\partial z} + d\frac{\partial^2}{\partial y^2} + 2e\frac{\partial^2}{\partial y\partial z} + f\frac{\partial^2}{\partial z^2} + \text{ lower order terms}$$

where the coefficients a, b, $\ldots$ are functions of (x, y, z). To L, we associate the symmetric matrix

$$M(x,y,z) = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Because M is symmetric, it has three real eigenvalues. Then, L is elliptic at a point (x_0, y_0, z_0) if all three eigenvalues of $M(x_0, y_0, z_0)$ are of the same sign; L is hyperbolic if two eigenvalues are of the same sign and the third of a different sign; and L is parabolic if one of the eigenvalues is 0.

4. Principle Of Superposition

Let L be a linear differential operator. The PDE $Lu = 0$ is said to be a *homogeneous* and the PDE $Lu = f$ (with $f \neq 0$) is said to be *nonhomogeneous.*

Claim. If u_1 and u_2 are solutions of the homogeneous equation $Lu = 0$, then for any constants c_1 and c_2, the linear combination $w = c_1u_1 + c_2u_2$ is also a solution of the homogeneous equation.

Proof. This is a direct consequence of the linearity of L. We have

$$Lw = L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2 = c_10 + c_20 = 0.$$

In general, we have the following

Principle of superposition 1. If $u_1, \cdots, u_N$ are solutions of the homogeneous equation $Lu = 0$, then their linear combination

$$w = c_1u_1 + \cdots + c_Nu_N = \sum_{j=1}^{N} c_ju_j, \qquad (c_1, \ldots, c_N, \text{ constants})$$

is also a solution of the homogeneous equation.

In general, the space of solutions of homogeneous PDEs contains infinitely many independent solutions and one might need to use not only a finite linear combination of solutions but an *infinite* linear combination of solutions. Thus one obtains an infinite series of functions. One is then tempted to conclude that the series of function obtained is again a solution of the homogeneous equation. This is indeed the case, if certain conditions are met. The series needs to converge to a twice differentiable function, and the termwise differentiation is allowed in the infinite series. For now, we state this as

Principle of superposition 1'. Suppose that

- $u_1,\ u_2,\ \cdots$ are infinitely many solutions of the homogeneous equation $Lu = 0$;
- the series $w = \sum_{j=1}^{\infty} c_ju_j$, (with $c_1,\ c_2,\ \ldots$ constants) converges to a twice differentiable function;
- term by term partial differentiation is valid for the series, i.e., $Dw = \sum c_jDu_j$, where D is any partial differentiation of order 1 or 2.

Then the function w given by the above series is again a solution of the homogeneous equation.

For the nonhomogeneous equation we have the following claim whose verification follows from the linearity of the operator.

Claim. If u_1 satisfies $Lu_1 = f_1$ and u_2 satisfies $Lu_2 = f_2$, then their linear combinations $w = c_1u_1 + c_2u_2$ satisfies

$$Lw = c_1f_1 + c_2f_2.$$

This leads to the following

Principle of superposition 2. If $u_1, \cdots, u_N$ are solutions of the nonhomogeneous equation $Lu_j = f_j$, then their linear combination

$$w = c_1 u_1 + \cdots + c_N u_N = \sum_{j=1}^{N} c_j u_j, \qquad (c_1, \ldots, c_N, \text{ constants})$$

is a solution of the equation

$$Lw = c_1 f_1 + \cdots + c_N f_N = \sum_{j=1}^{N} c_j f_j.$$

Of course, we can write a version 2' for this principle when we have infinitely many equations.

5. Decomposition Of BVP Into Simplers BVPs

In most physical and many mathematical problems, in addition to the PDE, there are conditions on the boundary that the solution must also satisfy these are the *boundary conditions.* The principle of superposition can be used to simplify the given BVP into simplers sub boundary value problems.

For example, consider a domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is the union of two curves Γ_1 and Γ_2 as in the figure.

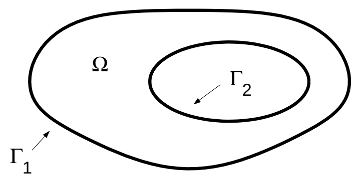

Figure 2. *Domain with boundary in* $\mathbb{R}^2$

Suppose that we have the BVP

$$\begin{cases} Lu = F & \text{inside the domain } \Omega \\ u = f_1 & \text{on the curve } \Gamma_1 \\ u = f_2 & \text{on the curve } \Gamma_2 \end{cases}$$

By using the principle of superposition, this problem can be decomposed as follows: This means we can find a solution u of the BVP as $u = v + w$, where v and w are the solution of the problems

$$\begin{cases} Lv = F & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_1 \\ v = 0 & \text{on } \Gamma_2 \end{cases} \quad \text{and} \quad \begin{cases} Lw = 0 & \text{in } \Omega \\ w = f_1 & \text{on } \Gamma_1 \\ w = f_2 & \text{on } \Gamma_2 \end{cases}$$

Let us verify that if v and w are solutions of the sub problems, then $u = v + w$ solves the original problem.

- The PDE: $Lu = Lv + Lw = F + 0 = F$;
- the boundary condition on Γ_1: $u = v + w = 0 + f_1 = f_1$;
- the boundary condition on Γ_2: $u = v + w = 0 + f_2 = f_2$.

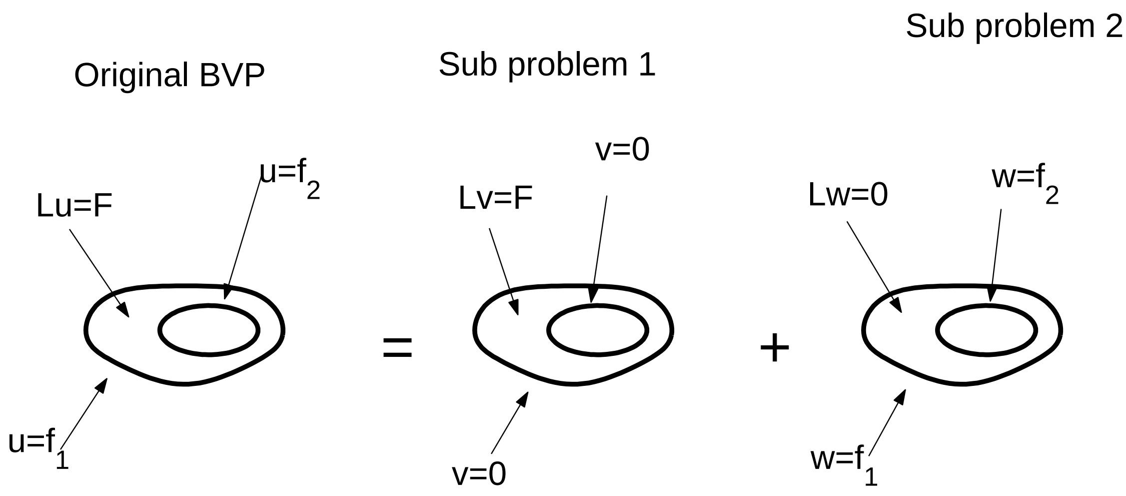

FIGURE 3. *Decomposition of a BVP into 2 sub problems*

Note that in the first sub problem the PDE in nonhomogeneous while the boundary conditions are homogeneous and in the second sub problem the PDE is homogeneous and the boundary conditions are nonhomogeneous.

The sub problem for w can be decomposed further into two sub problems as follows

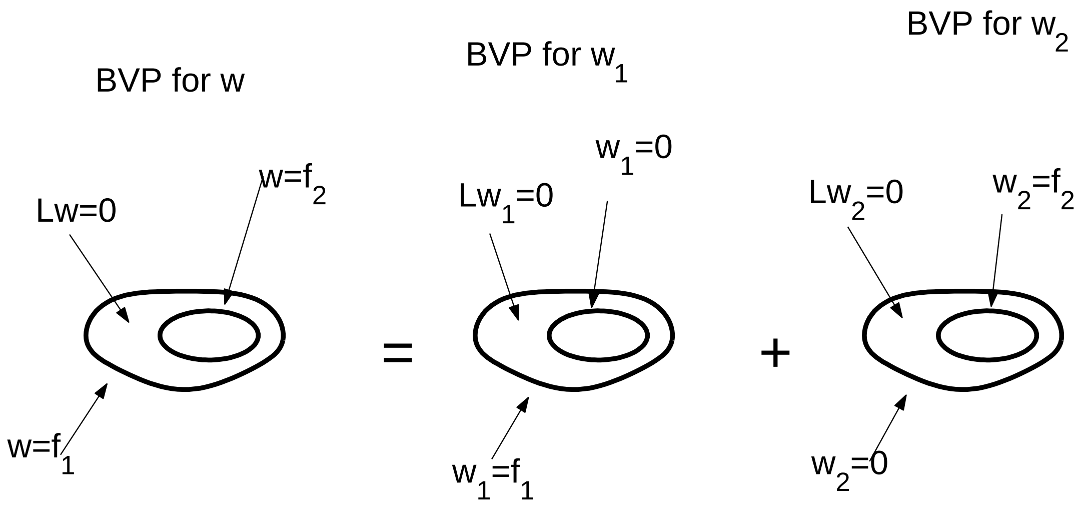

FIGURE 4. *Decomposition of the BVP for* w

$$\begin{cases} Lw_1 = 0 & \text{in } \Omega \\ w_1 = f_1 & \text{on } \Gamma_1 \\ w_1 = 0 & \text{on } \Gamma_1 \end{cases} \quad \text{and} \quad \begin{cases} Lw_2 = 0 & \text{in } \Omega \\ w_2 = 0 & \text{on } \Gamma_1 \\ w_2 = f_2 & \text{on } \Gamma_1 \end{cases}$$

To find the solution u of the original problem, we can find v, w_1, and w_2 (solutions of simpler problems) and set

$$u = v + w_1 + w_2.$$

6. Examples

6.1. **Example 1.** (Heat equation in a rod) Consider modeling heat conduction in a rod of length L. Assume that the initial temperature of the rod is given by a function $f(x)$, the left end is insulated and the right end is kept at a constant temperature of 100^0. The BVP is therefore

$$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & 0 < x < L, \quad t > 0; \\ u_x(0,t) = 0 & t > 0; \\ u(L,t) = 100 & t > 0; \\ u(x,0) = f(x) & 0 < x < L \end{cases}$$

This BVP can be decomposed as BVP1

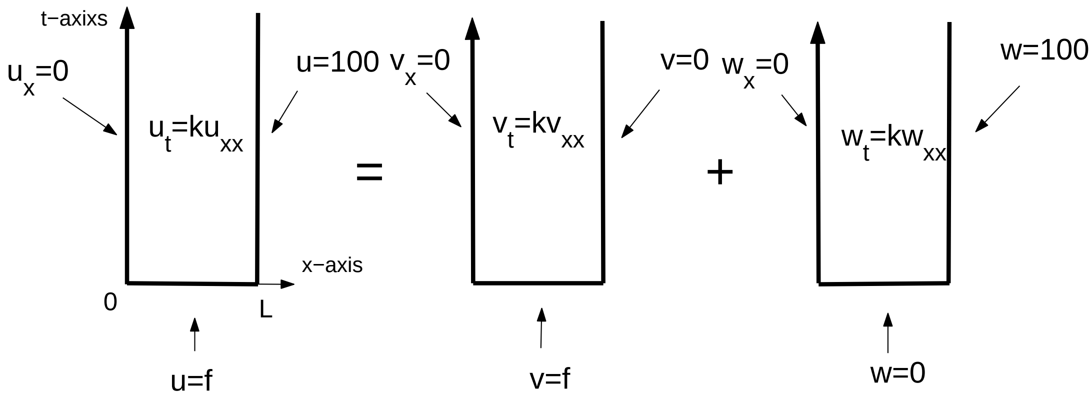

FIGURE 5. *Decomposition of the BVP in example 1*

$$\begin{cases} v_t(x,t) - k v_{xx}(x,t) = 0 & 0 < x < L, \quad t > 0; \\ v_x(0,t) = 0 & t > 0; \\ v(L,t) = 0 & t > 0; \\ v(x,0) = f(x) & 0 < x < L \end{cases}$$

and BVP2

$$\begin{cases} w_t(x,t) - k w_{xx}(x,t) = 0 & 0 < x < L, \quad t > 0; \\ w_x(0,t) = 0 & t > 0; \\ w(L,t) = 100 & t > 0; \\ w(x,0) = 0 & 0 < x < L \end{cases}$$

6.2. **Example 2.** (1-D Wave equation) Consider the small vibrations of a string of length L with fixed ends and with initial position and initial velocity given by the functions $f(x)$ and $g(x)$. The BVP is

$$\begin{cases} \Box u(x,t)=0 & 0<x<L, \quad t>0; \\ u(0,t)=0, & t>0; \\ u(L,t)=0, & t>0; \\ u(x,0)=f(x) & 0<x<L; \\ u_t(x,0)=g(x) & 0<x<L. \end{cases}$$

We can decompose this BVP (which contains two nonhomogeneous boundary conditions) into two subproblems, each of which contains only one single nonhomogeneous condition:

$$\text{BVP1} \quad \begin{cases} \Box v(x,t)=0; \\ v(0,t)=0; \\ v(L,t)=0; \\ v(x,0)=f(x); \\ v_t(x,0)=0. \end{cases} \qquad \text{BVP2} \quad \begin{cases} \Box w(x,t)=0; \\ w(0,t)=0; \\ w(L,t)=0; \\ w(x,0)=0; \\ w_t(x,0)=g(x). \end{cases}$$

Thus to find the solution u of the original problem, we can find separately v and w solutions of the simpler problems BVP1 and BVP2 and get $u=v+w$.

6.3. **Example 3.** (Dirichlet problem) Consider the following problem that models the steady-state temperature in a plate shaped like a quarter of a disk. Assume that the horizontal side is kept at 0^0, the vertical at 50^0, and the circular side at 100^0. The following figure shows how we can decompose the problem.

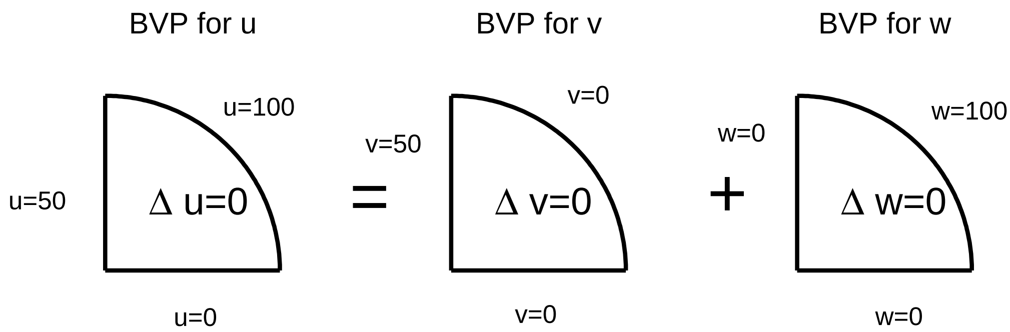

FIGURE 6. *Decomposition of the BVP in example 3*

7. Uniqueness of solutions of BVPs

We will be constructing solutions to BVPs and an important question that needs addressing is whether the constructed solution is the only solution or whether there other solutions. For the BVPs dealing with the heat, wave, and Laplace equations, the solution is indeed unique. We indicate why this is the case.

The uniqueness of the problems dealing with the Laplace operator Δ is based on a fundamental property satisfied by harmonic functions: *the maximum principle.* We state it as a Theorem

Theorem. (The Maximum Principle) *Suppose that u is a harmonic function (i.e. $\Delta u = 0$) in a domain $\Omega \subset \mathbb{R}^2$ (or in $\mathbb{R}^3$ or higher dimensional space). Suppose that*

Ω has a piecewise smooth boundary $\partial\Omega$ and that u is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$. Then the maximum (and minimum) values of u on $\overline{\Omega}$ occur on the boundary $\partial\Omega$. That is,

$$\max_{p\in\overline{\Omega}} u(p) = u(p_0) \quad \text{for some } p_0 \in \partial\Omega \, .$$

We also have

$$\min_{p\in\overline{\Omega}} u(p) = u(q_0) \quad \text{for some } q_0 \in \partial\Omega \, .$$

Now we illustrate how we can apply this property to show uniqueness of a BVP (Poisson problem)

Theorem. *Suppose that u is continuous on $\overline{\Omega}$ and is a solution of the BVP*

$$\begin{cases} \Delta u(p) = F(p) & p \in \Omega \\ u(p) = g(p) & p \in \partial\Omega \end{cases}$$

Then u is the **unique** *solution of the BVP*

Proof. Suppose that the BVP has two solutions u_1 and u_2, we need to show that $u_1 \equiv u_2$. Let $v = u_1 - u_2$. Then, by using the principle of superposition, we can prove that v satisfies the BVP

$$\begin{cases} \Delta v(p) = 0 & p \in \Omega \\ v(p) = 0 & p \in \partial\Omega \end{cases}$$

Thus v is a harmonic function in Ω that is identically zero on the boundary $\partial\Omega$. By the maximum principle, we have then $\max_{\overline{\Omega}} v = \min_{\overline{\Omega}} v = 0$. We deduce that $v \equiv 0$ and therefore $u_1 \equiv u_2$ (the two solutions are identical).

For problems dealing with the heat operator H there is also a version of the maximum principle that guarantees the uniqueness of the corresponding BVPs.

For the wave operator $\Box$, the uniqueness is proved by using energy conservation. We illustrate this idea for the vibrating string. Consider the BVP

$$(1) \qquad \begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, \ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < L \\ u_t(x,0) = g(x) & 0 < x < L \end{cases}$$

To a solution u of BVP (1), we associate its *energy integral* at time t as the function $E(t)$ defined by

$$E(t) = \frac{1}{2} \int_0^L \left(\frac{1}{c^2} u_t^2(x,t) + u_x^2(x,t)\right) dx$$

(E is the sum of the kinetic and potential energies). Although it appears that E depends on time t, it is in fact independent on t.

Lemma. *The function E is constant*

Proof. To prove that $E(t)$ is independent on t, we need to verify that $\dfrac{dE}{dt} \equiv 0$. We have

$$\begin{aligned}
\frac{dE}{dt} &= \int_0^L \left(\frac{u_{tt}u_t}{c^2} + u_{xt}u_x\right)dx && \text{since } (u_x^2)_t = 2u_{xt}u_x, \\
& && \text{and } (u_t^2)_t = 2u_{tt}u_t \\
&= \int_0^L (u_{xx}u_t + u_{xt}u_x)dx && \text{(replace } u_{tt} \text{ by } c^2u_{xx}) \\
&= [u_tu_x]_{x=0}^{x=L} - \int_0^L u_xu_{tx}dx + \int_0^L u_{xt}u_xdx && \text{(integ. by parts for } u_{xx}u_t) \\
&= u_t(L,t)u_x(L,t) - u_t(0,t)u_x(0,t)
\end{aligned}$$

Now it follows from $u(0,t) = u(L,t) = 0$ that

$$u_t(0,t) = u_t(L,t) = 0$$

This shows that $\dfrac{dE}{dt} \equiv 0$.

Now we are in position to prove the uniqueness for the solution of BVP (1).

Lemma. *If $u(x,t)$ is a solution of the BVP (1), continuous on the region $0 \leq x \leq L$, $t \geq 0$, then u is the unique solution of the BVP.*

Proof. Suppose that u_1 and u_2 are two solutions of (1). Let $v = u_1 - u_2$. By using the superposition principle, it is easy to see that the function v satisfies the following BVP

$$v_{tt} = c^2v_{xx}, \ v(0,t) = v(L,t) = 0, \ v(x,0) = 0, \ v_t(x,0) = 0.$$

We know from the previous Lemma that the energy of v is constant. Thus, $E(t) = E(0)$ for all $t \geq 0$. The energy of v at $t = 0$ is

$$E(0) = \frac{1}{2}\int_0^L \left(\frac{1}{c^2}v_t^2(x,0) + v_x^2(x,0)\right)dx = 0$$

since $v(x,0) = v_t(x,0) = 0$. Therefore,

$$E(t) = \frac{1}{2}\int_0^L \left(\frac{1}{c^2}v_t^2(x,t) + v_x^2(x,t)\right)dx = 0.$$

Note that the integrand in the last integral is the sum of two squares, the only way to have the integral identically 0 is if we have

$$v_x(x,t) = v_t(x,t) = 0 \qquad \forall x, t.$$

This in turn implies that $v(x,t) \equiv$ Constant. Finally, this constant is 0 since $v(x,0) = 0$.

8. Exercises

In Exercises 1 to 5, classify the PDE as either linear or nonlinear and give its order.

Exercise 1. $u_{xx} + u_{yy} = e^u$

Exercise 2. $3x^2u_x - 2(\ln y)u_y + \dfrac{3}{2x-1}u = 5$

Exercise 3. $u_{xy} = 1$

Exercise 4. $(u_x)^2 - 5u_y = 0$

Exercise 5. $u_{tt} - 2u_{xxxx} = \cos t$

In Exercises 6 to 10, classify each second order linear PDE with constant coefficients as either elliptic, parabolic, or hyperbolic in the plane $\mathbb{R}^2$.
Exercise 6. $u_{xy} = 0$

Exercise 7. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x - 12u_y + \sqrt{2}u = 0$

Exercise 8. $u_{xx} + 4u_{xy} + u_{yy} - 21u_y = \cos x$

Exercise 9. $2u_{xx} + 2\sqrt{2}u_{xy} + u_{yy} = \mathrm{e}^{xy}$

Exercise 10. $-u_{xx} + u_{xy} - u_{yy} + 3u = x^2$

In Exercises 11 to 13, find the regions in the (x, y)-plane where the second order PDE (with variable coefficients) is elliptic; parabolic; and hyperbolic.
Exercise 11. $x^2u_{xx} + u_{yy} = 0$

Exercise 12. $\sqrt{x^2+y^2}u_{xx} + 2u_{xy} + \sqrt{x^2+y^2}u_{yy} + xu_x + yu_y = 0$

Exercise 13. $u_{xx} + 2yu_{xy} + xu_{yy} - \cos(xy)u_y = 1$

Exercise 14. Verify that the functions $(x+1)\mathrm{e}^{-t}$, $\mathrm{e}^{-2x}\sin t$ and xt are, respectively solutions of the nonhomogeneous equations

$$Hu = -\mathrm{e}^{-t}(x+1), \quad Hu = \mathrm{e}^{-2x}(4\sin t + \cos t), \quad \text{and} \quad Hu = x$$

where H is the 1-D heat operator $H = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$

Find a solution of the PDE

$$Hu = \sqrt{2}x + \pi\mathrm{e}^{-2x}(4\sin t + \cos t) + \mathrm{e}^{-t}(x+1)$$

Exercise 15. Verify that the functions $x\cos(x-t)$ and $\sin(x+t) + \cos(\sqrt{2}t)$ are, respectively solutions of the nonhomogeneous equations

$$\Box u = 2\sin(x-t), \qquad \text{and} \qquad \Box u = -2\cos(\sqrt{2}t)$$

where $\Box$ is the 1-D wave operator $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$

Find a solution of the PDE

$$\Box u = -\sin(x-t) + \pi\cos(\sqrt{2}t)$$

Exercise 16. Verify that the functions r^2, $r^2\cos(2\theta)$ and $\sin(3\theta)$ are, respectively solutions of the PDEs

$$\Delta u = 2, \quad \Delta u = 0, \quad \text{and} \quad \Delta u = -\frac{9}{r^2}\sin(3\theta)$$

where Δ is the 2-D Laplace operator in polar coordinates $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$

Find a solution of the PDE

$$\Delta u = -1 + \sin(3\theta)$$

In Exercises 17 to 20, decompose the given BVP into simpler BVPs in such a way that only one nonhomogeneous condition appears in each sub BVP.

Exercise 17.

$$\begin{array}{ll} u_t - ku_{xx} = \cos t & 0 < x < L, \quad t > 0 \\ u(x,0) = 3x & 0 < x < L \\ u(0,t) = 0, \quad u(L,t) = 20 & t > 0 \end{array}$$

Exercise 18.

$$\begin{array}{ll} u_{tt} = 2u_{xx} + 2\sin x \cos t & 0 < x < \pi, \quad t > 0 \\ u(x,0) = \sin(3x), \quad u_t(x,0) = 1 & 0 < x < \pi \\ u(0,t) = \sin t, \quad u(\pi,t) = \cos t & t > 0 \end{array}$$

Exercise 19.

$$\begin{array}{ll} u_{xx} + u_{yy} = 5\cos x \sin y & 0 < x < \pi, \quad 0 < y < \pi \\ u(x,0) = 1, \quad u_y(x,\pi) = u(x,\pi) & 0 < x < \pi \\ u(0,y) = -1, \quad u_x(\pi,y) = -3u(\pi,y) & 0 < y < \pi \end{array}$$

Exercise 20.

$$\begin{array}{ll} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < r < 1, \quad 0 < \theta < \pi \\ u(r,0) = 10, \quad u(r,\pi) = 20, & 0 < r < 1 \\ u_r(1,\theta) = 5u(1,\theta) & 0 < \theta < \pi \end{array}$$

Exercise 21. Write the BVP for the steady-state temperature in a plate in the form of 30^0-sector of a ring with radii 1 and 2. One of the radial edges is kept at temperature 10^0C and on the other radial edge, the gradient of the temperature is numerically equal to the temperature. The outer circular edge is kept at temperature 100^0C, while the inner circular edge is insulated.

Decompose the BVP into Sub-BVPs that contain only one nonhomogeneous condition.