THE METHOD OF SEPARATION OF VARIABLES

To solve the BVPs that we have encountered so far, we will use *separation of variables* on the homogeneous part of the BVP. This separation of variables leads to problems for *ordinary differential equations* (some with endpoints conditions). The ODE problems are much easier to solve. (At this time you need to refresh your knowledge about linear differential equations that you have learned in the first course of differential equations: MAP2302).

To understand and appreciate the technique of separation of variables, we are going to solve in details some BVPs dealing with heat, wave , and Laplace equations. The nonhomogeneous terms of the problems below are cooked up so as to understand the situation. For more general nonhomogeneous terms, we have to wait until after Fourier series are studied.

The separation of variables is based on the following obvious observation,

Observation. *Let x and y be independent variables varying in intervals $I, J \subset \mathbb{R}$. Suppose that f and g are functions of x and y, respectively. If*

$$f(x) = g(y) \qquad \forall x \in I, \quad \forall y \in J,$$

then both functions are equal to the same constant. That is, there is a constant $C \in \mathbb{R}$ such that

$$f(x) = g(y) = C \qquad \forall x \in I, \quad \forall y \in J.$$

Example 1.

Consider the BVP modeling heat propagation in a rod of length L with both ends kept at constant temperature 0 degrees and the initial temperature of the rod is given by the function

$$f(x) = 100 \sin \frac{3\pi x}{L} - 50 \sin \frac{7\pi x}{L}.$$

If $u(x,t)$ denotes the temperature at time t of point x, then u satisfies the BVP

$$(1) \qquad \begin{cases} u_t(x,t) = k u_{xx}(x,t) & 0 < x < L, \quad t > 0 \; ; \\ u(0,t) = 0, \; u(L,t) = 0 & t > 0 \; ; \\ u(x,0) = f(x) & 0 < x < L \; . \end{cases}$$

where k is the thermal diffusivity of the rod. Now we describe the method of separation for this BVP.

Step 1. *Separate the BVP into its homogeneous part (HP) and nonhomogeneous part (NHP).* In this problem the HP is:

$$(2) \qquad \begin{cases} u_t(x,t) = k u_{xx}(x,t) & 0 < x < L, \quad t > 0 \; ; \\ u(0,t) = 0, \; u(L,t) = 0 & t > 0 \; . \end{cases}$$

and the NHP is

$$(3) \qquad u(x,0) = f(x) \quad 0 < x < L \; .$$

Date: February 3, 2016.

Note that the homogeneous part HP consists of all equations that are satisfied by the zero function. So in our BVP, $u(x,t) \equiv 0$ solves the first three equations but not the fourth. Hence, the first three equations form HP and the fourth is NHP.

Step 2. *Separate HP into ODE problems.* Assume that HP has a *nontrivial*[1] solution of the form

$$u(x,t) = X(x)T(t)$$

with X a function of x alone and T a function of t alone. Such a function has separated variables. We rewrite the HP for such a function u. It becomes

$$\begin{cases} X(x)T'(t) = kX''(x)T(t) & 0 < x < L, \quad t > 0 \;; \\ X(0)T(t) = 0, \;\; X(L)T(t) = 0 & t > 0 \;. \end{cases} \tag{4}$$

At all points where $X(x)T(t) \neq 0$ the PDE can be written as

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

The functions T'/kT and X''/X depend on the independent variables t and x and they are equal. Thus the functions are equal to a constant:

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda \;,$$

with λ a constant (called the separation constant). The minus sign in front of λ is just for latter convenience. It follows that X and T solve the ODEs

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < L, \quad \text{and} \quad T'(t) + k\lambda T(t) = 0, \quad t > 0^2 \;.$$

The boundary conditions in (4) imply that $X(0) = 0$ and $X(L) = 0$ (since $T \not\equiv 0$). We have therefore split HP into two ODE problems. The X-problem:

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < L \;; \\ X(0) = X(L) = 0 \;, \end{cases} \tag{5}$$

and the T-problem:

$$T' + k\lambda T = 0 \;, \quad t > 0 \;. \tag{6}$$

The X-problem, which consists of a second order ode and two endpoints conditions, is an example of a *Sturm-Liouville problem.* The values of λ for which the SL-problem has nontrivial solutions are called the *eigenvalues* of the problem and the corresponding nontrivial solutions are the *eigenfunctions.*

Step 3. *Find the eigenvalues and eigenfunctions of the SL-problem.* The characteristic equation of the ODE in the SL-problem is

$$m^2 + \lambda = 0 \quad \text{with roots} \quad m = \pm\sqrt{-\lambda} \;.$$

Depending on the sign of λ, we distinguish three cases:

Case $\imath$: $\lambda < 0$. We write $\lambda = -\nu^2$ with $\nu > 0$. The characteristic roots are $m = \pm\nu$. The general solution of the ODE in the X-problem is

$$X(x) = Ae^{\nu x} + Be^{-\nu x} \;,$$

[1] By nontrivial solutions we mean the solutions that are not identically zero.

[2] To be more rigorous, we have these odes in the regions where $XT \neq 0$ but then it follows from uniqueness that these equations are valid on the intervals where X and T are defined.

with A and B constants. We need to find A and B so that X satisfies the endpoints conditions:

$$\begin{array}{lcl} X(0) = 0 & \Rightarrow & A + B = 0 \\ X(L) = 0 & \Rightarrow & A\mathrm{e}^{L\nu} + B\mathrm{e}^{-L\nu} = 0 \end{array}$$

The above linear system for A and B has the unique solution $A = B = 0$. The reason is the following. From the first equation we have $B = -A$ and then the second equation becomes

$$A(\mathrm{e}^{L\nu} - \mathrm{e}^{-L\nu}) = 0 \ .$$

Now $L > 0$ and $\nu > 0$ imply that $\mathrm{e}^{L\nu} - \mathrm{e}^{-L\nu} > 0$. Hence the equation for A holds only when $A = 0$.

With $A = 0$ and $B = 0$, the function X is zero $X(x) = 0$ for $0 \le x \le L$. We have proved that if $\lambda < 0$, the only solution for the X-problem is $X \equiv 0$. This means that $\lambda < 0$ cannot be an eigenvalue of the SL-problem.

Case *ii***:** $\lambda = 0$. The characteristic roots are $m = 0$ (with multiplicity 2). The general solution of the ODE of the X-problem is

$$X(x) = A + Bx \ ,$$

with A, B constants. The endpoints conditions $X(0) = X(L) = 0$ imply that A and B satisfy the system

$$A = 0 \quad \text{and} \quad A + BL = 0$$

whose only solution is $A = 0$, $B = 0$. The corresponding solution of the X-problem is again $X \equiv 0$. Therefore, $\lambda = 0$ is not an eigenvalue.

Case *iii***:** $\lambda > 0$. We write $\lambda = \nu^2$ with $\nu > 0$. The characteristic roots are $m = \pm i\nu$. The general solution of the ODE in the X-problem is

$$X(x) = A\cos(\nu x) + B\sin(\nu x) \ ,$$

with A and B constants. The endpoint conditions implies

$$A\cos 0 + B\sin 0 = 0, \qquad A\cos(L\nu) + B\sin(L\nu) = 0 \ .$$

The first equation gives $A = 0$ and the second reduces to

$$B\sin(L\nu) = 0 \ .$$

Note that if $B = 0$, then again $X \equiv 0$ is the trivial solution. Thus in order to have a nontrivial solution X, we must have

$$\sin(L\nu) = 0 \ .$$

Therefore, ν must be of the form

$$\nu = \frac{n\pi}{L} \ , \quad \text{with} \quad n = 1,\ 2,\ 3,\ \ldots$$

In this case

$$X(x) = B\sin\frac{n\pi x}{L}$$

is a nontrivial solution of the X-problem. This solution is obtained when

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \ ,$$

which is therefore an eigenvalue of the SL-problem.

Summary of the above discussion. The SL-problem has infinitely many eigenvalues. For each $n \in \mathbb{Z}^+$ we have:

$$\text{eigenvalue}: \ \lambda_n = \left(\frac{n\pi}{L}\right)^2 , \qquad \text{eigenfunction}: \ X_n(x) = \sin\frac{n\pi x}{L} .$$

Note that any nonzero multiple of X_n is also an eigenfunction. We need only write a generator of the eigenspace.

Step 4. *For each eigenvalue, solve the corresponding T-problem.* For $\lambda = \lambda_n$, the T-problem is:

$$T'(t) + k\left(\frac{n\pi}{L}\right)^2 T(t) = 0 .$$

The general solution is

$$T(t) = C\mathrm{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{with } C \text{ constant.}$$

Step 5. *Use steps 3 and 4 to write the solutions with separated variables for the Homogeneous problem HP:* For each $n \in \mathbb{Z}^+$, we have:

- eigenvalue: $\lambda_n = (n\pi/L)^2$;
- eigenfunction: $X_n(x) = \sin(n\pi x/L)$;
- T-solution: $T_n(t) = \exp(-k(n\pi/L)^2 t)$;

a corresponding solution of HP with separated variable is:

$$u_n(x,t) = T_n(t)X_n(x) = \mathrm{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L} .$$

Step 6. *Use the principle of superposition to obtain more solutions of HP.* Any linear combination of the solutions u_n obtained in step 5 is again a solution of problem (2). That is, for $c_1, \cdots, c_N$ constants the function

$$u(x,t) = c_1 u_1(x,t) + \cdots + c_N u_N(x,t) = \sum_{j=1}^{N} c_j \mathrm{e}^{-k\left(\frac{j\pi}{L}\right)^2 t} \sin\frac{j\pi x}{L}$$

is also a solution of (2). In fact, under appropriate conditions, we can form a linear combination of infinitely many solutions u_n and obtain again a solution of (2) (see Principle of Superposition 1').

Step 7. *Find the coefficients c_j so that the solution of HP found in step 6 satisfies also the* **nonhomogeneous condition** *(3).* This means, we need to find $c_1, \ c_2, \ \cdots$ so that

$$u(x,0) = f(x) = 100\sin\frac{3\pi x}{L} - 50\sin\frac{7\pi x}{L}.$$

We have,

$$u(x,0) = c_1\sin\frac{\pi x}{L} + c_2\sin\frac{2\pi x}{L} + \cdots + c_N\sin\frac{N\pi x}{L} = 100\sin\frac{3\pi x}{L} - 50\sin\frac{7\pi x}{L}$$

We take all coefficients c_j to be zero except $c_3 = 100$ and $c_7 = -50$.

Conclusion: The solution of the original BVP (1) is:

$$u(x,t) = 100\mathrm{e}^{-k(3\pi/L)^2 t}\sin\frac{3\pi x}{L} - 50\mathrm{e}^{-k(7\pi/L)^2 t}\sin\frac{7\pi x}{L} .$$

The Exponential and the Trigonometric Functions

Recall that if $x \in \mathbb{R}$, then

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x \, .$$

The trigonometric functions cos and sin can be expressed as:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \, .$$

For $x, y \in \mathbb{R}$, we have $e^{x+iy} = e^{x}(\cos y + i \sin y)$.

The hyperbolic trigonometric functions cosh (cosine hyperbolic) and sinh (sine hyperbolic) are defined by

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^{x} - e^{-x}}{2} \, .$$

We also have

$$e^{x} = \cosh x + \sinh x \quad \text{and} \quad e^{-x} = \cosh x - \sinh x \, .$$

The following identities of cosh and sinh are left for you to verify as an exercise.

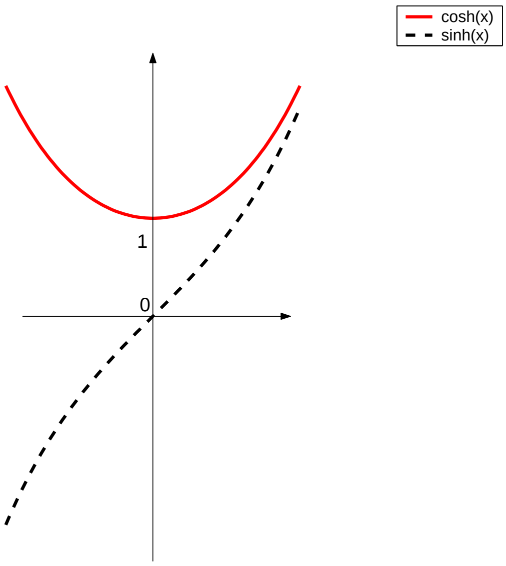

Figure 1. *Graphs of* cosh *and* sinh

- $\cosh' x = \sinh x$; $\sinh' x = \cosh x$;
- $\cosh'' x = \cosh x$; $\sinh'' x = \sinh x$.
- $\cosh(x + y) = \cosh x \, \cosh y + \sinh x \, \sinh y$;
- $\sinh(x + y) = \sinh x \, \cosh y + \cosh x \, \sinh y$;
- $\cosh(x - y) = \cosh x \, \cosh y - \sinh x \, \sinh y$;
- $\sinh(x - y) = \sinh x \, \cosh y - \cosh x \, \sinh y$;
- $\cosh^2 x - \sinh^2 y = 1$;

Example 2.

The following BVP models wave propagation with *damping* in a string of length L.

$$\text{(7)} \qquad \begin{cases} u_{tt}(x,t) + 2au_t(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, \quad t > 0\,; \\ u(0,t) = 0, \ u(L,t) = 0 & t > 0\,; \\ u(x,0) = f(x), \ u_t(x,0) = g(x) & 0 < x < L\,. \end{cases}$$

The wave's speed is c and the damping constant is a. We assume that $0 \le a < c\pi/L$. This problem models the vibrations in a string of length L with end points attached. The initial position and initial velocity are given by the functions $f(x)$ and $g(x)$. Take for instance,

$$\text{(8)} \qquad f(x) = 20\sin\frac{\pi x}{L} - \sin\frac{6\pi x}{L}, \quad \text{and} \quad g(x) = 3\sin\frac{\pi x}{L} + \sin\frac{13\pi x}{L}.$$

To solve this problem, we proceed with steps of Example 1.

Step 1. *Separate the BVP into its homogeneous part (HP) and nonhomogeneous part (NHP).*

Homogeneous part (HP):

$$\text{(9)} \qquad \begin{cases} u_{tt}(x,t) + 2au_t(x,t) = c^2 u_{xx}(x,t) & 0 < x < L, \quad t > 0\,; \\ u(0,t) = 0, \ u(L,t) = 0 & t > 0\,. \end{cases}$$

Nonhomogeneous part (NHP)

$$\text{(10)} \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad 0 < x < L\,.$$

Step 2. *Separate HP into ODE problems.* Assume that $u(x,t) = X(x)T(t)$ is a nontrivial solution of (HP) We rewrite the HP for such a function u. It becomes

$$\text{(11)} \qquad \begin{cases} X(x)T''(t) + 2aX(x)T'(t) = c^2 X''(x)T(t) & 0 < x < L, \quad t > 0\,; \\ X(0)T(t) = 0, \ X(L)T(t) = 0 & t > 0\,. \end{cases}$$

By separating the variables, the PDE can be written as

$$\frac{T''(t)}{c^2T(t)} + \frac{2aT'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \text{ (constant)}.$$

It follows that X and T solve the ODEs

$$X''(x) + \lambda X(x) = 0\,, \quad \text{and} \quad T''(t) + 2aT'(t) + c^2\lambda T(t) = 0\,.$$

The boundary conditions imply that $X(0) = 0$ and $X(L) = 0$. Problem (11) splits then into two ODE problems. The X-problem (SL-problem):

$$\text{(12)} \qquad \begin{cases} X'' + \lambda X = 0 & 0 < x < L\,; \\ X(0) = X(L) = 0\,, \end{cases}$$

and the T-problem:

$$\text{(13)} \qquad T'' + 2aT' + c^2\lambda T = 0\,, \quad t > 0\,.$$

Step 3. *Find the eigenvalues and eigenfunctions of the SL-problem.* The X-problem was solved in Example 1 and we found that it has infinitely many eigenvalues. For each $n \in \mathbb{Z}^+$ we have:

$$\begin{array}{ll} \text{eigenvalue :} & \lambda_n = \nu_n^2 = \left(\dfrac{n\pi}{L}\right)^2\,, \\ \text{eigenfunction :} & X_n(x) = \sin\nu_n x = \sin\dfrac{n\pi x}{L}\,. \end{array}$$

Step 4. *For each eigenvalue, solve the corresponding T-problem.* For $\lambda = \lambda_n$, the T-problem is:

$$T''(t) + 2aT'(t) + c^2\nu_n^2 T(t) = 0 \ .$$

The characteristic equation of this ODE is

$$m^2 + 2am + (c\nu_n)^2 = 0$$

with roots

$$m = -a \pm \sqrt{a^2 - (c\nu_n)^2} \ .$$

Note that since $a < c\pi/L$, then $a^2 - (c\nu_n)^2 < 0$ for every $n \in \mathbb{Z}^+$. We can then write $a^2 - (c\nu_n)^2 = -\omega_n^2$ with $\omega_n > 0$. Thus, the characteristic roots are complex conjugate and can be written as

$$m = -a \pm i\omega_n \ .$$

Two independent solutions of the T-problem are:

$$T_n^1(t) = \mathrm{e}^{-at}\cos\omega_n t \quad \text{and} \quad T_n^2(t) = \mathrm{e}^{-at}\sin\omega_n t \ .$$

Step 5. *Use steps 3 and 4 to write the solutions with separated variables for the Homogeneous problem (11).* For each $n \in \mathbb{Z}^+$, we have:

- eigenvalue: $\lambda_n = \nu_n^2 = (n\pi/L)^2$;
- eigenfunction: $X_n(x) = \sin(\nu_n x) = \sin(n\pi x/L)$;
- T-solutions: $T_n^1(t) = \mathrm{e}^{-at}\cos\omega_n t$ and $T_n^2(t) = \mathrm{e}^{-at}\sin\omega_n t$

Corresponding solutions of HP with separated variable are:

$$\begin{aligned} u_n^1(x,t) &= T_n^1(t)X_n(x) = \mathrm{e}^{-at}\cos(\omega_n t)\sin(\nu_n x) \ ; \\ u_n^2(x,t) &= T_n^2(t)X_n(x) = \mathrm{e}^{-at}\sin(\omega_n t)\sin(\nu_n x) \ . \end{aligned}$$

Step 6. *Use the principle of superposition to obtain more solutions of HP.* Any linear combination of the solutions u_n^1 and u_n^2 obtained in step 5 is again a solution of problem (9). That is, for $A_1, \cdots, A_N$ and $B_1, \cdots, B_N$ constants the function

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{N} A_j u_j^1(x,t) + B_j u_j^2(x,t) \\ &= \sum_{j=1}^{N} \mathrm{e}^{-at}\left[A_j\cos(\omega_j t) + B_j\sin(\omega_j t)\right]\sin(\nu_j x) \ . \end{aligned}$$

is also a solution of problem (9).

Step 7. *Find the coefficients A_j and B_j so that the solution of HP found in step 6 satisfies also the nonhomogeneous conditions (10).* This means, we need to find $A_1,\ B_1,\ A_2,\ B_2,\ \cdots$ so that

$$u(x,0) = f(x) = 20\sin\frac{\pi x}{L} - \sin\frac{6\pi x}{L} \ ; \tag{14}$$

$$u_t(x,0) = g(x) = 3\sin\frac{\pi x}{L} + \sin\frac{13\pi x}{L}. \tag{15}$$

We need to write $u_t(x,0)$. We have the following

$$u_t(x,t) = \sum_{j=1}^{N} \mathrm{e}^{-at}\left[(B_j\omega_j - aA_j)\cos(\omega_j t) - (A_j\omega_j + aB_j)\sin(\omega_j t)\right]\sin(\nu_j x)$$

The initial position and initial conditions (14) and (15) became

$$u(x,0)=\sum_{j=1}^{N} A_j \sin\frac{j\pi x}{L} = 20\sin\frac{\pi x}{L} - \sin\frac{6\pi x}{L}$$
$$u_t(x,0)=\sum_{j=1}^{N}(B_j\omega_j - aA_j)\sin\frac{j\pi x}{L} = 3\sin\frac{\pi x}{L} + \sin\frac{13\pi x}{L}.$$

By equating the coefficients, the first equation gives

$$A_1 = 20,\quad A_6=-1,\quad A_j = 0 \ \text{ if } j\neq 1 \ \text{ and } \ j\neq 6\ .$$

The second equation gives

$$B_1\omega_1 - aA_1 = 3,\quad B_{13}\omega_{13} - aA_{13} = 1,\quad B_j\omega_j - aA_j = 0 \ \text{ if } \ j\neq 1,\ 13\ .$$

Thus,

$$B_1 = \frac{3+20a}{\omega_1},\ B_{13} = \frac{1}{\omega_{13}},\ B_6 = \frac{-a}{\omega_6},\ \text{and } B_j = 0,\ j\neq 1,\ 6,\ 13.$$

Conclusion: The solution of the original BVP (7) is:

$$\begin{aligned} u(x,t) &= \mathrm{e}^{-at}\left[20\cos(\omega_1 t) + \frac{3+20a}{\omega_1}\sin(\omega_1 t)\right]\sin(\nu_1 x) + \\ &+\mathrm{e}^{-at}\left[-\cos(\omega_6 t) - \tfrac{a}{\omega_6}\sin(\omega_6 t)\right]\sin(\nu_6 x) + \tfrac{1}{\omega_{13}}\mathrm{e}^{-at}\sin(\omega_{13}t)\sin(\nu_{13}x)\ . \end{aligned}$$

EXAMPLE 3.

Consider the BVP given in polar coordinates by

$$\left\{\begin{array}{ll} \Delta u(r,\theta) = 0\ , & 1<r<2\,,\ 0\le\theta\le 2\pi\ ; \\ u(1,\theta) = \cos\theta\ , & 0\le\theta\le 2\pi\ ; \\ u_r(2,\theta) = \sin 2\theta;\,, & 0\le\theta\le 2\pi\ . \end{array}\right. \tag{16}$$

This BVP can be interpreted as that for the steady-state temperature in an annular

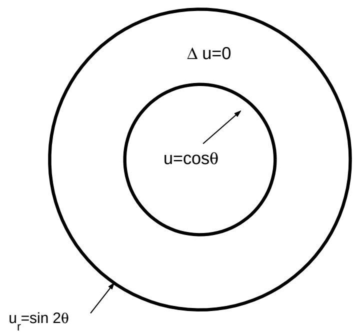

FIGURE 2. *BVP with mixed boundary conditions*

region where on the outer boundary the heat flux is prescribed while on the inner boundary the temperature is prescribed. Recall that $\Delta u = 0$ in polar coordinates means

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Remark The polar coordinates (r,θ) and $(r,\theta+2\pi)$ represent the same point in the plane. So any function u defined in the plane is 2π-periodic in θ: $u(r,\theta) = u(r,\theta+2\pi)$. Hence, although it is not explicitly listed in the BVP, it is understood that $u(r,0) = u(r,2\pi)$ and also $u_\theta(r,0) = u_\theta(r,2\pi)$.

Now, we proceed as in the previous problems.

Step 1. *Separate the BVP into its homogeneous part (HP) and nonhomogeneous part (NHP).*

Homogeneous part (HP):

$$u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta} = 0 \quad 1 < r < 2, \;\; 0 \le \theta \le 2\pi \; . \tag{17}$$

Nonhomogeneous part (NHP)

$$u(1,\theta) = \cos\theta, \;\; u_r(2,\theta) = \sin 2\theta, \quad 0 \le \theta \le 2\pi \; . \tag{18}$$

Step 2. *Separate HP into ODE problems.* Assume that $u(r,\theta) = R(r)\Theta(\theta)$ is a nontrivial solution of (HP). We rewrite HP for such a function u. It becomes

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0 \tag{19}$$

By separating the variables, the PDE can be written as

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\frac{r^2R''(r)}{R(r)} - \frac{rR'(r)}{R(r)} = -\lambda \quad \text{(constant)}.$$

It follows that Θ and R solve the ODEs

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \; , \quad \text{and} \quad r^2R''(r) + rR'(r) - \lambda R(r) = 0 \; .$$

Remember that u and also u_θ must be 2π-periodic in θ. Thus the functions $\Theta(\theta)$ and $\Theta'(\theta)$ must be 2π-periodic:

$$\Theta(0) = \Theta(2\pi), \qquad \Theta'(0) = \Theta'(2\pi)$$

We have then the following ODE problems. The Θ-problem (periodic SL-problem):

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \quad 0 \le \theta \le 2\pi \; ; \\ \Theta(0) = \Theta(2\pi) \quad \text{and} \quad \Theta'(0) = \Theta'(2\pi) \; . \end{cases} \tag{20}$$

and the R-problem:

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \; , \quad 1 < r < 2 \; . \tag{21}$$

Step 3. *Find the eigenvalues and eigenfunctions of the periodic SL-problem.* The characteristic equation of the ODE in the SL-problem is

$$m^2 + \lambda = 0 \quad \text{with roots} \quad m = \pm\sqrt{-\lambda} \; .$$

Depending on the sign of λ, we distinguish three cases:

Case $\imath$: $\lambda < 0$. We write $\lambda = -\nu^2$ with $\nu > 0$. The characteristic roots are $m = \pm\nu$. The general solution of the ODE in the Θ-problem is

$$\Theta(\theta) = A\cosh(\nu x) + B\sinh(\nu x) \; ,$$

with A and B constants. We need to find A and B so that Θ satisfies the periodic conditions:

$$\begin{array}{lcl} \Theta(0) = \Theta(2\pi) & \Rightarrow & A = A\cosh(2\pi\nu) + B\sinh(2\pi\nu) \\ \Theta'(0) = \Theta'(2\pi) & \Rightarrow & B\nu = A\nu\sinh(2\pi\nu) + B\nu\cosh(2\pi\nu) \end{array}$$

The above linear system for A and B can be written as

$$\begin{aligned}(\cosh(2\pi\nu)-1)A+\sinh(2\pi\nu)B &= 0\\ \sinh(2\pi\nu)A+(\cosh(2\pi\nu)-1)B &= 0\end{aligned}$$

The determinant of this linear system is

$$(\cosh(2\pi\nu)-1)^2-\sinh^2(2\pi\nu)=2(1-\cosh(2\pi\nu))\neq 0$$

(because $\nu>0$ and $\cosh(x)=1$ only when $x=0$). Since the determinant of the linear system is not zero, the unique solution is $A=B=0$. Hence $\Theta\equiv 0$. Thus $\lambda<0$ is not an eigenvalue.

Case *ii*: $\lambda=0$. The characteristic roots are $m=0$ (with multiplicity 2). The general solution of the ODE of the Θ-problem is

$$\Theta(\theta)=A+B\theta\ ,$$

with $A,\ B$ constants. The periodic conditions are

$$A=A+2\pi B \quad \text{and} \quad B=B.$$

We get $B=0$ and A arbitrary. Therefore $\lambda=0$ is an eigenvalue with eigenfunction $\Theta\equiv 1$.

Case *iii*: $\lambda>0$. We write $\lambda=\nu^2$ with $\nu>0$. The characteristic roots are $m=\pm i\nu$. The general solution of the ODE in the Θ-problem is

$$\Theta(\theta)=A\cos(\nu\theta)+B\sin(\nu\theta)\ ,$$

with A and B constants. The periodic conditions give

$$A=A\cos(2\pi\nu)+B\sin(2\pi\nu),\qquad \nu B=-\nu A\sin(2\pi\nu)+\nu B\cos(2\pi\nu)\ .$$

We rewrite this system in standard form

$$\begin{aligned}(1-\cos(2\pi\nu))A-\sin(2\pi\nu)B &= 0\\ \sin(2\pi\nu)A+(1-\cos(2\pi\nu))B &= 0\end{aligned}$$

The determinant of this system is $2(1-\cos(2\pi\nu))$. In order to have nontrivial solutions, it is necessary to have the determinant equal to 0. That is,

$$\cos(2\pi\nu)=1\ .$$

For this to happen, we need $\nu=1,\ 2,\ 3,\ \cdots$. When $\nu=n\in\mathbb{N}$, any values of A and B satisfy the system. Hence, for $\lambda=n^2$, problem (20) has nontrivial solutions: $\cos(n\theta)$ and $\sin(n\theta)$.

Summary: The eigenvalues and eigenfunctions of the Θ-problem are

Eigenvalues	Eigenfunctions
$\lambda_0=0$	$\Theta_0(\theta)=1$
$\lambda_n=n^2$, $(n\in\mathbb{Z}^+)$	$\Theta_n^1(\theta)=\cos(n\theta)$ and $\Theta_n^2(\theta)=\sin(n\theta)$

Step 4. *For each eigenvalue, solve the corresponding R-problem.* Note that the ode for R is a *Cauchy-Euler* equation. We can find solutions of the form r^m. The characteristic equation is

$$m(m-1)+m-\lambda=0 \quad\Rightarrow\quad m^2-\lambda=0\ .$$

For $\lambda=0$, we have $m=0$ with multiplicity 2 and the general solution of the ode is

$$R(r)=A+B\ln r\ .$$

For $\lambda = \lambda_n = n^2$, the characteristic roots are $m = \pm n$ and the general solution of the ode is

$$R(r) = Ar^n + \frac{B}{r^n}$$

Step 5. *Use steps 3 and 4 to write the solutions with separated variables for the Homogeneous problem (19).*

For $\lambda = 0$, we have

$$u_0(r, \theta) = (A_0 + B_0 \ln r)$$

For $\lambda = n^2$, with $n \in \mathbb{Z}^+$, we have

$$u_n^1(r, \theta) = \left(A_n^1 r^n + \frac{B_n^1}{r^n} \right) \cos(n\theta), \quad u_n^2(r, \theta) = \left(A_n^2 r^n + \frac{B_n^2}{r^n} \right) \sin(n\theta)$$

Step 6. Any linear combination of the solutions obtained in step 5:

$$u(x, t) = u_0(r, \theta) + \sum_{n=1}^{N} u_n^1(r, \theta) + u_n^2(r, \theta)$$

is also a solution of problem (19).

Step 7. Now we seek the coefficients A_j^1, A_j^2, B_j^1 and B_j^2 so that the solution u of step 6 satisfies the nonhomogeneous conditions $u(1, \theta) = \cos\theta$ and $u_r(2, \theta) = \sin 2\theta$. The first condition gives (check this claim)

$$\begin{array}{l} A_0 = 0, \quad A_n^2 + B_n^2 = 0, \; n = 1, \cdots, N \\ A_1^1 + B_1^1 = 1, \quad A_n^1 + B_n^1 = 0, \; n = 2, \cdots, N \end{array}$$

The second condition gives

$$\begin{array}{l} \dfrac{B_0}{2} = 0, \quad A_n^1 2^{n-1} - \dfrac{B_n^1}{2^{n+1}} = 0, \; n = 1, \cdots, N \\ 2(A_2^2 2 - \dfrac{B_2^2}{2^3}) = 1, \quad A_n^2 2^{n-1} - \dfrac{B_n^2}{2^{n+1}} = 0, \; n = 1, 3, 4, \cdots, N \end{array}$$

It gives

$$A_0 = B_0 = 0, \; A_n^1 = B_n^1 = 0 \text{ for } n \neq 1, \; A_j^2 = B_j^2 = 0 \text{ for } j \neq 2$$

and

$$A_1^1 = \frac{1}{5}, \quad B_1^1 = \frac{4}{5}, \quad A_2^2 = \frac{2}{17}, \quad B_2^2 = -\frac{2}{17}$$

The solution of the original problem (16) is

$$u(r, \theta) = \frac{1}{5}\left(r + \frac{4}{r} \right) \cos\theta + \frac{2}{17}\left(r^2 - \frac{1}{r^2} \right) \sin(2\theta)$$

Example 4

Consider the BVP

$$(22) \qquad \begin{cases} u_t(x, t) = u_{xx}(x, t) \; , & 0 < x < \pi, \quad t > 0 \; ; \\ u(0, t) = 0 \; , & t > 0 \; ; \\ u(\pi, t) = u_x(\pi, t) \; , & t > 0 \; ; \\ u(x, 0) = f(x) & 0 < x < \pi \; . \end{cases}$$

This problem models heat conduction in a rod in which one end (at $x = 0$) is kept at 0 degrees and at the other end ($x = \pi$) the heat flux is controlled. The initial temperature is $f(x)$.

Step 1. The homogeneous part is

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & 0 < x < \pi, \quad t > 0\ ; \\ u(0,t) = 0,\ u(\pi,t) = u_x(\pi,t) & t > 0\ . \end{cases} \tag{23}$$

and the nonhomogeneous part is

$$u(x,0) = f(x) \quad 0 < x < \pi\ . \tag{24}$$

Step 2. Separation of variables for the homogeneous part. Suppose $u(x,t) = X(x)T(t)$ solves the homogeneous problem (23). Then

$$\begin{cases} X(x)T'(t) = X''(x)T(t) & 0 < x < \pi, \quad t > 0\ ; \\ X(0)T(t) = 0,\ X(\pi)T(t) = X'(\pi)T(t) & t > 0\ . \end{cases} \tag{25}$$

The separation of variables leads to the ODE problems for X and T. The X-problem:

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < \pi\ ; \\ X(0) = 0 \quad X(\pi) = X'(\pi)\ , \end{cases} \tag{26}$$

and the T-problem:

$$T' + \lambda T = 0\ , \quad t > 0\ . \tag{27}$$

Step 3. Eigenvalues and eigenfunctions of the X-problem. The characteristic equation of the ODE in the SL-problem is

$$m^2 + \lambda = 0 \quad \text{with roots} \quad m = \pm\sqrt{-\lambda}\ .$$

Depending on the sign of λ, we distinguish three cases:

Case $\imath$**:** $\lambda < 0$. Set $\lambda = -\nu^2$ with $\nu > 0$. The characteristic roots are $m = \pm\nu$. The general solution of the ODE in the X-problem is

$$X(x) = A\mathrm{e}^{\nu x} + B\mathrm{e}^{-\nu x}\ ,$$

with A and B constants. The endpoint conditions implies

$$\begin{aligned} &A + B = 0, \\ &A\mathrm{e}^{\pi\nu} + B\mathrm{e}^{-\pi\nu} = \nu[A\mathrm{e}^{\pi\nu} - B\mathrm{e}^{-\pi\nu}]\ . \end{aligned}$$

In order to get a nontrivial solution ν must satisfy the equation

$$\mathrm{e}^{\nu\pi} - \mathrm{e}^{-\nu\pi} = \nu[\mathrm{e}^{\nu\pi} + \mathrm{e}^{-\nu\pi}]$$

or equivalently

$$\mathrm{e}^{2\nu\pi} = \frac{1+\nu}{1-\nu}$$

This equation has a unique positive solution ν_0. Furthermore $\nu_0 \in (0,\ 1)$ (check this claim as an exercise). Thus $\lambda_0 = -\nu_0^2$ is the unique negative eigenvalue with the corresponding eigenfunction

$$X_0(x) = \frac{\mathrm{e}^{\nu_0 x} - \mathrm{e}^{-\nu_0 x}}{2} = \sinh(\nu_0 x)$$

Case $\imath\imath$**:** $\lambda = 0$. I leave it as an exercise for you to show that 0 cannot be an eigenvalue.

Case *iii*: $\lambda > 0$. We write $\lambda = \nu^2$ with $\nu > 0$. The characteristic roots are $m = \pm i\nu$. The general solution of the ODE in the X-problem is

$$X(x) = A\cos(\nu x) + B\sin(\nu x) \ ,$$

with A and B constants. The endpoint conditions implies

$$\begin{aligned} &A\cos 0 + B\sin 0 = 0, \\ &A\cos(\pi\nu) + B\sin(\pi\nu) = -\nu A\sin(\pi\nu) + \nu B\cos(\pi\nu) \ . \end{aligned}$$

The first equation gives $A = 0$ and the second reduces to

$$B\sin(\pi\nu) = B\nu\cos(\pi\nu) \ .$$

Thus in order to have a nontrivial solution X, the parameter $\nu > 0$ we must satisfy

$$\sin(\pi\nu) = \nu\cos(\pi\nu) \ \Rightarrow \ \tan(\pi\nu) = \nu \ . \tag{28}$$

This equation has infinitely many positive solutions. For each $k \in \mathbb{Z}^+$, it has a

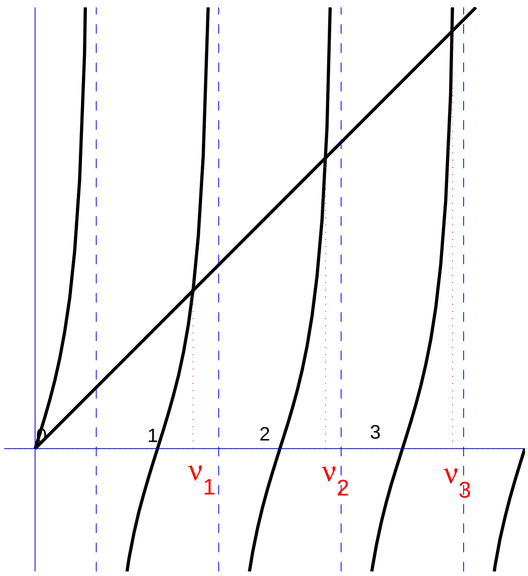

FIGURE 3. *Solutions of equation* $\tan(\pi\nu) = \nu$

unique solution ν_k between k and $k + (1/2)$. To see why, consider the function

$$f(\nu) = \sin(\pi\nu) - \nu\cos(\pi\nu) \ .$$

We have $f(k) = -k(-1)^k$ and $f(k + (1/2)) = (-1)^k$. These values have opposite signs. It follows from the intermediate value theorem that there is at least on number $\nu_k \in (k, \ k + (1/2))$ at which $f(\nu_k) = 0$. To show that there is only one solution between k and $k + (1/2)$, we can look at the derivative of f:

$$f'(\nu) = (\pi - 1)\cos(\pi\nu) + \pi\nu\sin(\pi\nu) \ .$$

Note that $\cos(\pi\nu)$ and $\sin(\pi\nu)$ are either both positive or both negative in the interval $(k, \ k + (1/2))$. Hence, f' does not change sign in the interval and f is either increasing or either decreasing in the interval. Therefore, it cannot have two zeroes. I will leave it as an exercise to verify that f does not have zeroes between $k+(1/2)$ and $k+1$. We can also consider the second form of the equation $\tan(\pi\nu) = \nu$. The solutions are given by the intersection of the graphs of the functions $y = \tan(\pi\nu)$ and $y = \nu$ (See figure).

Therefore, to each solution $\nu_j > 0$ of the above equation $\tan(\pi\nu) = \nu$ corresponds a nontrivial solution $X_j(x) = \sin(\nu_j x)$ of the Sturm-Liouville problem (26). A numerical estimate of the first 5 values ν_j and $\lambda_j = \nu_j^2$ is given in the following table.

n	1	2	3	4	5
ν_n	1.2901	2.3731	3.4092	4.4293	5.4422
$\lambda_n = \nu_n^2$	1.6644	5.6314	11.6225	19.6189	29.6171

Step 4. For $\lambda = \lambda_n = \nu_n^2$, the T-problem is:

$$T'(t) + \nu_n^2 T(t) = 0 \ .$$

The general solution is

$$T(t) = C\mathrm{e}^{-\nu_n^2 t} \quad \text{with } C \text{ constant.}$$

For $\lambda = \lambda_0 = -\nu_0^2$. The T-problem is:

$$T'(t) - \nu_0^2 T(t) = 0 \ .$$

The general solution is

$$T(t) = C\mathrm{e}^{\nu_0^2 t} \quad \text{with } C \text{ constant.}$$

Step 5. For each $n \in \mathbb{Z}^+$, a solution with separated variable of the homogeneous part (23) is

$$u_n(x,t) = T_n(t)X_n(x) = \mathrm{e}^{-\nu_n^2 t}\sin(\nu_n x) \ .$$

and for $n = 0$ we have

$$u_0(x,t) = T_0(t)X_0(x) = \mathrm{e}^{\nu_0^2 t}\sinh(\nu_0 x) \ .$$

Step 6. Any linear combination of the solutions u_n is again a solution of problem (23). That is, for $c_0, c_1, \cdots$, constants the function

$$u(x,t) = c_0\mathrm{e}^{\nu_0^2 t}\sinh(\nu_0 x) + c_1\mathrm{e}^{-1.6644\,t}\sin(1.2901x) + c_2\mathrm{e}^{-5.6314\,t}\sin(2.3731x) + \cdots$$

solves (23).

Step 7. If we know how to express $f(x)$ in terms of the eigenfunctions $\sin(\nu_n x)$ and $\sinh(\nu_0 x)$, then we would match the coefficients. To complete this step, we need to study Fourier series.

Example 5

Consider the BVP

(29)
$$\begin{cases} u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t)\,, & 0<x<\pi,\ \ 0<y<2\pi \ \ t>0\,; \\ u(x,y,0)=0\,, & 0<x<\pi,\ \ 0<y<2\pi\,; \\ u_t(x,y,0) = \sin x \sin(7y/2)\,, & 0<x<\pi,\ \ 0<y<2\pi\,; \\ u(x,0,t)=u(x,2\pi,t)=0\,, & 0<x<\pi \ \ t>0\,; \\ u(0,y,t)=u(\pi,y,t)=0\,, & 0<y<2\pi\,,\ \ t>0. \end{cases}$$

This problem models transversal vibrations of a rectangular membrane with fixed boundary. In this example, we have three independent variables x, y, and t.

Step 1. The homogeneous part is

(30)
$$\begin{cases} u_{tt}(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t)\,, & 0<x<\pi,\ \ 0<y<2\pi \ \ t>0\,; \\ u(x,y,0)=0\,, & 0<x<\pi,\ \ 0<y<2\pi\,; \\ u(x,0,t)=u(x,2\pi,t)=0\,, & 0<x<\pi \ \ t>0\,; \\ u(0,y,t)=u(\pi,y,t)=0\,, & 0<y<2\pi\,,\ \ t>0. \end{cases}$$

and the nonhomogeneous part is

$$u_t(x,y,0) = \sin x \sin\frac{7y}{2}\,,\quad 0<x<\pi,\ \ 0<y<2\pi\,. \tag{31}$$

Step 2. Separation of variables for the homogeneous part. Suppose $u(x,y,t) = X(x)Y(y)T(t)$ solves the homogeneous problem (30). Then

(32)
$$\begin{cases} XYT'' = X''YT + XY''T\,, & 0<x<\pi,\ \ 0<y<2\pi \ \ t>0\,; \\ X(x)Y(y)T(0)=0\,, & 0<x<\pi,\ \ 0<y<2\pi\,; \\ X(x)Y(0)T(t) = X(x)Y(2\pi)T(t)=0\,, & 0<x<\pi \ \ t>0\,; \\ X(0)Y(y)T(t)=X(\pi)Y(y)T(t)=0\,, & 0<y<2\pi\,,\ \ t>0. \end{cases}$$

We separate the PDE as follows

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda \quad \text{(constant)}.$$

Then

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - \lambda = -\alpha \quad \text{(constant)}.$$

If we let $\beta = \lambda - \alpha$, then we get

$$\frac{X''(x)}{X(x)} = -\alpha\,,\quad \frac{Y''(y)}{Y(y)} = -\beta\,,\quad \frac{T''(t)}{T(t)} = -\lambda\,,\quad \lambda = \alpha+\beta$$

The boundary conditions translate into

$$X(0)=X(\pi)=0,\quad Y(0)=Y(2\pi)=0,\quad T(0)=0$$

The ODE problems for X, Y and T are:

$$\begin{array}{ll} X''+\alpha X = 0\,, & X(0)=X(\pi)=0 \\ Y''+\beta Y=0\,, & Y(0)=Y(2\pi)=0 \\ T''+\lambda T=0\,, & T(0)=0\,. \end{array} \tag{33}$$

with $\lambda = \alpha+\beta$. The X-problem and Y-problems are independent SL-problems.

Step 3. Eigenvalues and eigenfunctions of the X-problem and the Y-problem.

For the X-problem we have $\forall j \in \mathbb{Z}^+$, the eigenvalue and eigenfunction are

$$\alpha_j = j^2, \qquad X_j(x) = \sin(jx) \ .$$

For the Y-problem we have $\forall k \in \mathbb{Z}^+$, the eigenvalue and eigenfunction are

$$\beta_k = \left(\frac{k}{2}\right)^2, \qquad Y_k(y) = \sin\frac{ky}{2} \ .$$

Step 4. For each pair of eigenvalues α_j, β_k, solve the T-problem. We have

$$\lambda_{jk} = \alpha_j + \beta_k = j^2 + \frac{k^2}{4} = \frac{4j^2+k^2}{4} = \omega_{jk}^2$$

The corresponding general solution of the ode for T is

$$T = A\cos(\omega_{jk}t) + B\sin(\omega_{jk}t).$$

The initial condition $T(0) = 0$ gives $A = 0$.

Step 5. For each $j \in \mathbb{Z}^+$ and each $k \in \mathbb{Z}^+$, a solution with separated variable of the homogeneous part (30) is

$$u_{jk}(x,y,t) = T_{jk}(t)X_j(x)Y_k(y) = \sin(\omega_{jk}t)\sin(jx)\sin\frac{ky}{2}$$

with $\omega_{jk} = \dfrac{\sqrt{4j^2+k^2}}{2}$.

Step 6. Any linear combination of the solutions u_{jk} is again a solution of problem (30). That is, any function of the form

$$u(x,y,t) = \sum_{j\le N,\ k\le M} c_{jk}\sin(\omega_{jk}t)\sin(jx)\sin\frac{ky}{2}$$

solves problem (30).

Step 7. Finally, we need to find the coefficients c_{jk} so that the above function solves the nonhomogeneous condition. We have

$$u_t(x,y,t) = \sum_{j\le N,\ k\le M} c_{jk}\omega_{jk}\cos(\omega_{jk}t)\sin(jx)\sin\frac{ky}{2}$$

Hence,

$$u_t(x,y,0) = \sum_{j\le N,\ k\le M} c_{jk}\omega_{jk}\sin(jx)\sin\frac{ky}{2} = \sin x\sin\frac{7y}{2}$$

Therefore, $c_{jk} = 0$ for all j,k except when $j = 1$ and $k = 7$, in which case

$$\omega_{1,7}c_{1,7} = 1 \ \Rightarrow\ c_{1,7} = \frac{1}{\omega_{1,7}} = \frac{2}{\sqrt{53}} \ .$$

The solution of the original problem (29) is

$$u(x,y,t) = \frac{2}{\sqrt{53}}\sin\left(\frac{\sqrt{53}}{2}t\right)\sin x\sin\frac{7y}{2} \ .$$

Exercises

Use the method of separation of variables to solve the following boundary value problems.

Exercise 1.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\pi\,, \quad t > 0 \\ u(0,t) = u(2\pi,t) = 0 & t > 0 \\ u(x,0) = \sin\dfrac{x}{2} - \sin(3x) & 0 < x < 2\pi \end{array}$$

Exercise 2.

$$\begin{array}{ll} u_t(x,t) = u_{xx}(x,t) & 0 < x < 2\pi\,, \quad t > 0 \\ u_x(0,t) = u_x(2\pi,t) = 0 & t > 0 \\ u(x,0) = 100 - 3\cos\dfrac{5x}{4} & 0 < x < 2\pi \end{array}$$

Exercise 3.

$$\begin{array}{ll} u_t(x,t) = 2u_{xx}(x,t) & 0 < x < \pi\,, \quad t > 0 \\ u(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = 50\sin\dfrac{x}{2} - 25\sin\dfrac{7x}{2} + 10\sin\dfrac{11x}{2} & 0 < x < \pi \end{array}$$

Exercise 4.

$$\begin{array}{ll} u_{tt}(x,t) = \sqrt{2}u_{xx}(x,t) & 0 < x < 3\pi\,, \quad t > 0 \\ u(0,t) = u(3\pi,t) = 0 & t > 0 \\ u_t(x,0) = 0 & 0 < x < 3\pi \\ u(x,0) = f(x) & 0 < x < 3\pi \end{array}$$

where

$$f(x) = \sin\frac{x}{3} - \frac{1}{2}\sin(2x) + \frac{1}{5}\sin\frac{7x}{3}$$

Exercise 5.

$$\begin{array}{ll} u_{tt}(x,t) = \sqrt{2}u_{xx}(x,t) & 0 < x < 3\pi\,, \quad t > 0 \\ u(0,t) = u(3\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < 3\pi \\ u_t(x,0) = g(x) & 0 < x < 3\pi \end{array}$$

where

$$g(x) = \sin\frac{2x}{3} + \frac{1}{2}\sin\frac{5x}{3}$$

Exercise 6.

$$\begin{array}{ll} u_{tt}(x,t) = u_{xx}(x,t) & 0 < x < 1\,, \quad t > 0 \\ u(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = \sin\dfrac{\pi x}{2} & 0 < x < 1 \\ u_t(x,0) = -\sin\dfrac{7\pi x}{2} & 0 < x < 1 \end{array}$$

Exercise 7.

$$\begin{aligned}
&u_{tt}(x,t) = u_{xx}(x,t) && 0 < x < 1\,, \quad t > 0\\
&u_x(0,t) = u_x(1,t) = 0 && t > 0\\
&u(x,0) = 0 && 0 < x < 1\\
&u_t(x,0) = 1 - \frac{1}{2}\cos\frac{5\pi x}{2} && 0 < x < 1
\end{aligned}$$

Exercise 8.

$$\begin{aligned}
&\Delta u(x,y) = 0 && 0 < x < 1\,, \quad 0 < y < 2\\
&u(0,y) = u(1,y) = 0 && 0 < y < 2\\
&u(x,0) = \sin(2\pi x), \quad u(x,2) = \sin(3\pi x) && 0 < x < 1
\end{aligned}$$

Exercise 9.

$$\begin{aligned}
&\Delta u(x,y) = 0 && 0 < x < 1\,, \quad 0 < y < 2\\
&u(0,y) = 1 - \cos\frac{\pi y}{2}, \quad u(1,y) = 3\cos\frac{5\pi y}{2} && 0 < y < 2\\
&u_y(x,0) = u_y(x,2) = 0 && 0 < x < 1
\end{aligned}$$

Exercise 10. (In polar coordinates)

$$\begin{aligned}
&\Delta u(r,\theta) = 0 && 0 < r < 2\,, \quad 0 \le \theta \le 2\pi\\
&u(2,\theta) = 1 + \cos(3\theta) - 2\sin(5\theta) && 0 \le \theta \le 2\pi
\end{aligned}$$

Exercise 11. (In polar coordinates)

$$\begin{aligned}
&\Delta u(r,\theta) = 0 && 1 < r < 2\,, \quad 0 \le \theta \le 2\pi\\
&u(1,\theta) = \sin(3\theta) && 0 \le \theta \le 2\pi\\
&u_r(2,\theta) = \cos(5\theta) && 0 \le \theta \le 2\pi
\end{aligned}$$

Exercise 12.

$$\begin{aligned}
&u_t(x,y,t) = \Delta u(x,y,t) && 0 < x < \pi\,,\ 0 < y < 2\pi\,,\ t > 0\\
&u(x,0,t) = u(x,2\pi,t) = 0 && 0 < x < \pi\,,\ t > 0\\
&u(0,y,t) = u(\pi,y,t) = 0 && 0 < y < 2\pi\,,\ t > 0\\
&u(x,y,0) = \sin x \sin\frac{3y}{2} && 0 < x < \pi\,,\ 0 < y < 2\pi
\end{aligned}$$

Exercise 13.

$$\begin{aligned}
&u_t(x,y,t) = \Delta u(x,y,t) && 0 < x < \pi\,,\ 0 < y < 2\pi\,,\ t > 0\\
&u(x,0,t) = u(x,2\pi,t) = 0 && 0 < x < \pi\,,\ t > 0\\
&u_x(0,y,t) = u_x(\pi,y,t) = 0 && 0 < y < 2\pi\,,\ t > 0\\
&u(x,y,0) = \sin 5x \sin y && 0 < x < \pi\,,\ 0 < y < 2\pi
\end{aligned}$$

Exercise 14.

$$\begin{aligned}
&u_{tt}(x,y,t) = \sqrt{2}\Delta u(x,y,t) && 0 < x < \pi\,,\ 0 < y < 2\pi\,,\ t > 0\\
&u_y(x,0,t) = u_y(x,2\pi,t) = 0 && 0 < x < \pi\,,\ t > 0\\
&u(0,y,t) = u(\pi,y,t) = 0 && 0 < y < 2\pi\,,\ t > 0\\
&u(x,y,0) = \sin 3x \cos\frac{5y}{2} && 0 < x < \pi\,,\ 0 < y < 2\pi\\
&u_t(x,y,0) = 0 && 0 < x < \pi\,,\ 0 < y < 2\pi
\end{aligned}$$

In exercises 15 to 17 find all solutions with separated variables.

Exercise 15.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\,, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(2,t) = -u_x(2,t) & t > 0 \end{array}$$

Exercise 16.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\,, \quad t > 0 \\ u_x(0,t) = 0 & t > 0 \\ u(2,t) = -u_x(2,t) & t > 0 \end{array}$$

Exercise 17.

$$\begin{array}{ll} u_t(x,t) = 3u_{xx}(x,t) & 0 < x < 2\,, \quad t > 0 \\ u(0,t) = u_x(0,t) & t > 0 \\ u(2,t) = u_x(2,t) & t > 0 \end{array}$$