FOURIER SERIES PART I: DEFINITIONS AND EXAMPLES

To a 2π-periodic function $f(x)$ we will associate a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) ,$$

or in terms of the exponential e^{ix}, a series of the form

$$\sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{inx} .$$

For most of the functions that we will be dealing with, these series are in a sense equal to f. Before we do so, we need some terminology and introduce a class of functions.

1. Piecewise continuous and piecewise smooth functions

We will be using the following notation:

$$\begin{array}{lll} x \to c^+ & \text{means} & x \to c \text{ and } x > c , \\ x \to c^- & \text{means} & x \to c \text{ and } x < c . \end{array}$$

For a function $f(x)$ we will use the following

$$f(c^+) = \lim_{x \to c^+} f(x) \quad \text{and} \quad f(c^-) = \lim_{x \to c^-} f(x)$$

provided that the above limits exist and are finite.

A function f is said to be *piecewise continuous* on the closed interval $[a,\ b]$ if there exit finitely many points

$$c_0 = a < c_1 < c_2 < \cdots < c_{n-1} < c_n = b$$

such that:

- f is continuous in each open interval $(c_{j-1},\ c_j)$, for $j = 1, \cdots, n$;
- $f(a^+)$, $f(b^-)$ exist and
- for $j = 1, \cdots, n$, the limits $f(c_j^-)$ and $f(c_j^+)$ exist.

Hence, f is continuous everywhere except possibly at a finite number of points where it has *jump* discontinuities. The space of piecewise continuous functions on $[a,\ b]$ will be denoted by $C_p^0[a,\ b]$.

Example 1. Consider the function

$$f(x) = \begin{cases} \dfrac{x}{2} + 1 & \text{if } -1 < x < 0 ; \\ x^2 - 1 & \text{if } 0 < x < 1 ; \\ 0 & \text{if } 1 < x \le 2 \end{cases}$$

Then $f \in C_p^0[-1,\ 2]$. Inside the interval, the only discontinuities of f are at 0 and

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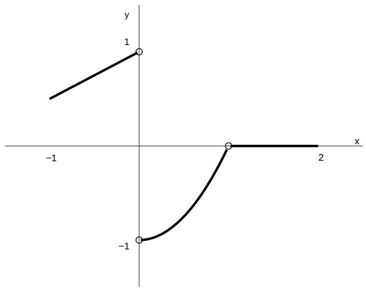

FIGURE 1. *Piecewise continuous*

1. We have

$$f(-1^+) = .5, \quad f(0^-) = 1, \quad f(0^+) = -1, \quad f(1^-) = f(1^+) = 0, \quad f(2^-) = 0.$$

Example 2. Consider the function

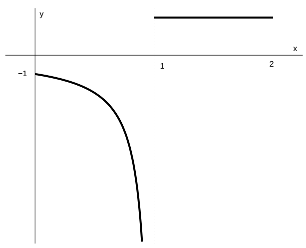

FIGURE 2. *Not piecewise continuous*

$$f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } 0 \leq x < 1 \ ; \\ 2 & \text{if } 1 \leq x \leq 2 \ . \end{cases}$$

The inside discontinuity of f is at $x = 1$. Because $\lim_{x\to 1^-} f(x) = -\infty$ (not finite), then $f \notin C_p^0([0,\ 2]$.

A function f is said to be *piecewise smooth* on a closed interval $[a,\ b]$ if f is piecewise continuous on $[a,\ b]$ and f' is also piecewise continuous on $[a,\ b]$. This means that $f'(x)$ exists and is continuous everywhere in $(a,\ b)$ except possibly at a finite number of points. Moreover, at each point where f' is discontinuous the one sided limits of f' exist and are finite. The space of piecewise continuous on $[a,\ b]$ will be denoted by $C_p^1[a,\ b]$.

Example 3. The function $f(x) = |x|$ is piecewise smooth on $[-1,\ 5]$. It is continuous on $[-1,\ 5]$ and its derivative exists everywhere except at $x = 0$. At 0, we have $f'(0+) = 1$ and $f'(0-) = -1$.

Example 4. The function

$$f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \ ; \\ (x-1)^{2/3} & \text{if } 0 < x < 2 \ . \end{cases}$$

is piecewise continuous but it is not piecewise smooth. The function is piecewise

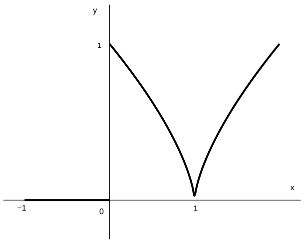

FIGURE 3. *Piecewise continuous but not piecewise smooth*

continuous because $f(-1^+) = 0$, $f(2^-) = 1$, and at $x = 0$, it has a jump discontinuity ($f(0^-) = 0$ and $f(0^+) = 1$). The derivative f' is

$$f'(x) = 0 \quad \text{for } -1 < x < 0 \quad \text{and} \quad f'(x) = \frac{2}{3(x-1)^{1/3}} \quad \text{for } 0 < x < 2, \;\; x \neq 1$$

The derivative f' is not piecewise continuous because $f'(1^{\pm})$ are not finite (the function f has a cusp at $x = 1$).

A function f is said to be piecewise continuous (respectively piecewise smooth) on the whole real line $\mathbb{R}$ if f is piecewise continuous (resp. piecewise smooth) on each closed interval $[a, b] \subset \mathbb{R}$.

Remark. Note that if $f \in C_p^0[a, b]$, then f is integrable on $[a, b]$. That is, $\displaystyle\int_a^b f(x)dx$ is a finite number. Indeed, with $c_1 < c_2 < \cdots < c_{n-1}$ the jump discontinuities of f, we have

$$\int_a^b f(x)dx = \int_{c_0}^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)dx$$

and each integral on the right is finite because within the limits of integration, the function is bounded and continuous.

2. Even and odd functions

A function f is said to be an *even* (respectively *odd*) function if

$$f(-x) = f(x) \;\; (\text{resp. } f(-x) = -f(x)) \qquad \forall x \text{ in domain of } f.$$

Note that it follows from the definition that the domain of an even or an odd

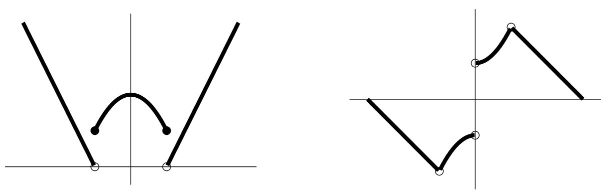

FIGURE 4. *Graphs of even and odd functions*

function needs to de symmetric with respect to $0 \in \mathbb{R}$. That is, if x is in the domain, then $(-x)$ is also in the domain. The graph of an even function is symmetric with respect to the y-axis and the graph of an odd function is symmetric with respect to the origin. When integrating even or odd functions, it is useful to use the following property

Lemma. *If $f \in C_p^0[-A,\ A]$ is an even function, then*

$$\int_{-A}^{A} f(x)dx = 2\int_{0}^{A} f(x)dx \ .$$

If $f \in C_p^0[-A,\ A]$ is an odd function, then

$$\int_{-A}^{A} f(x)dx = 0 \ .$$

Proof. For an even function f, we have

$$\begin{aligned}\int_{-A}^{A} f(x)dx &= \int_{-A}^{0} f(x)dx + \int_{0}^{A} f(x)dx \\ &= \int_{A}^{0} f(-u)d(-u) + \int_{0}^{A} f(x)dx \quad \text{(substitution } u=-x) \\ &= \int_{0}^{A} f(u)du + \int_{0}^{A} f(x)dx \quad \text{(use } f \text{ even)} \\ &= 2\int_{0}^{A} f(x)dx\end{aligned}$$

An analogous argument can be applied when f is odd.

Remark. I leave it as an exercise for you to prove that the product of two even functions is even; two odd functions is even; and the product of an even and odd function is odd. Thatis

Even $\times$ Even = Even
Odd $\times$ Odd = Even
Even $\times$ Odd = Odd

3. Periodic functions

Recall that a function f is said to be *periodic* with period T ($T > 0$) if

$$f(x+T) = f(x-T) = f(x) \quad \forall x \text{ in domain of } f \ .$$

This implies in particular that if x is in the domain of f, then $x + kT$ is also in the domain of f for every integer k and $f(x + kT) = f(x)$.

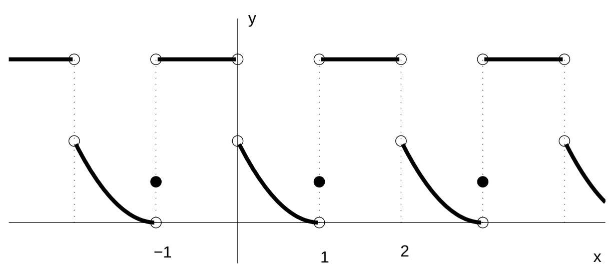

Figure 5. *Graph of a periodic function, $T = 2$*

Example 1. For example, the functions $\sin x$ and $\cos x$ are 2π-periodic and $\tan x$ is π-periodic. In general, if ω is constant, then $\sin(\omega x)$ and $\cos(\omega x)$ have period $T = 2\pi/\omega$.

Example 2. Denote by [ ] the greatest integer function. Thus, for $x \in \mathbb{R}$, $[x]$ the greatest integer less or equal than x. For instance, $[1.7] = 1$, $[\sqrt{5}] = 2$, $[\pi] = 3$,

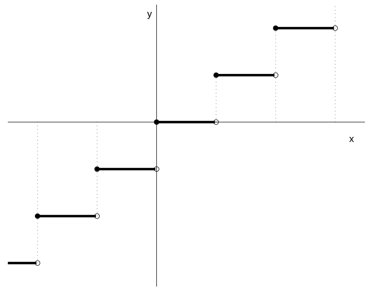

FIGURE 6. *The step function* $[x]$

$[-0.001] = -1$, $[-\sqrt{7}] = -3$, etc. Note that [ ] satisfies the property $[x+1] = [x]+1$ for ever $x \in \mathbb{R}$.

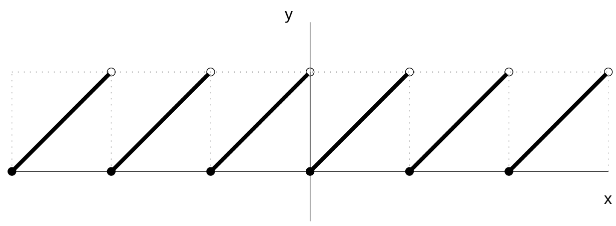

FIGURE 7. *The sawtooth function:* $x - [x]$

The function f defined by $f(x) = x - [x]$ (the fractional part of x), also called the *sawtooth function*, is piecewise smooth in $\mathbb{R}$ and periodic with period $T = 1$

$$f(x+1) = (x+1) - [x+1] = x + 1 - ([x]+1) = x - [x] = f(x), \quad \forall x \in \mathbb{R} .$$

Example 3. (Triangular wave function) I leave it as an exercise for you to check that the function f defined by

$$f(x) = \left| x - 2\left[\frac{x+1}{2}\right] \right|$$

is continuous and periodic with period $T = 2$.

An important property of periodic functions is the following.

Theorem *Let f be piecewise continuous in $\mathbb{R}$ and periodic with period T. Then*

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx , \qquad \forall a \in \mathbb{R}$$

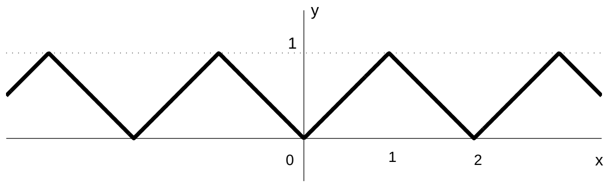

FIGURE 8. *The triangular wave function*

Proof. Consider the function $H(a)$ defined for $a \in \mathbb{R}$ by $H(a) = \int_a^{a+T} f(x)dx$. To prove the Theorem, we need to show that H is constant. We rewrite H as

$$H(a) = \int_a^0 f(x)dx + \int_0^{a+T} f(x)dx = \int_0^{a+T} f(x)dx - \int_0^a f(x)dx \ .$$

We can compute the derivative $H'(a)$ by using the *Fundamental Theorem of Calculus* and find

$$H'(a) = f(a+T) - f(a) = 0$$

because f is T-periodic. $H' \equiv 0$ gives H constant.

4. Orthogonality of functions

We define an inner product $< \ , \ >$ in the space $C_p^0[a,\ b]$ of piecewise continuous functions on $[a,\ b]$ by

$$< f, g >= \int_a^b f(x)g(x)dx \ , \quad \text{for } f, g \in C_p^0[a,\ b]$$

The *norm* of a function $f \in C_p^0[a,\ b]$ is defined

$$||f|| = \sqrt{< f, f >} = \left(\int_a^b f(x)^2 dx \right)^{1/2} \ .$$

Example 1. Let $f(x) = |x|$, $g(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \end{cases}$, and $h(x) = \dfrac{x}{|x|}$. Note that $h(x) = 1$ for $x > 0$ and $h(x) = -1$ for $x < 0$. In $C_p^0[-1,\ 1]$, we have

$$< f, g >= \int_{-1}^1 f(x)g(x)dx = \int_{-1}^0 (-x)dx + \int_0^1 2xdx = \frac{1}{2} + 1 = \frac{3}{2}$$
$$< f, h >= \int_{-1}^1 |x|\frac{x}{|x|}dx = \int_{-1}^1 xdx = 0$$
$$< g, h >= \int_{-1}^0 (-1)dx + \int_0^1 2dx = 1$$
$$||f|| = \left(\int_{-1}^1 |x|^2 dx \right)^{1/2} = \sqrt{\frac{2}{3}}$$
$$||g|| = \left(\int_{-1}^0 dx + \int_0^1 4dx \right)^{1/2} = \sqrt{5}$$
$$||h|| = \left(\int_{-1}^1 dx \right)^{1/2} = \sqrt{2}$$

Two functions $f, g \in C_p^0[a,\ b]$ are said to be *orthogonal* if $< f, g >= 0$. That is, f and g are orthogonal if

$$\int_a^b f(x)g(x)dx = 0\ .$$

A set of functions $S \subset C_p^0[a,\ b]$ is *orthonormal* if the elements of S are mutually orthogonal and each element of S has norm 1. That is,

- $< f, g >= 0$ for every $f, g \in S,\ f \neq g$
- $||f|| = 1$ for every $f \in S$.

Example 2. The functions $f(x) = x - 1$ and $g(x) = (x-1)^2$ are orthogonal in $C_p^0[0,\ 2]$ since,

$$< f, g >= \int_0^2 (x-1)(x-1)^2 dx = \frac{1}{4}\ (x-1)^4\big|_{x=0}^{x=2} = 0.$$

The set $S = \{f, g\}$ is orthogonal but it is not orthonormal because

$$||f|| = \left(\int_0^2 (x-1)^2 dx\right)^{1/2} = \sqrt{\frac{2}{3}}$$
$$||g|| = \left(\int_0^2 (x-1)^4 dx\right)^{1/2} = \sqrt{\frac{2}{5}}$$

However, if we replace f and g by

$$f_1(x) = \frac{f(x)}{||f||} = \sqrt{\frac{3}{2}}\, f(x) \quad \text{and} \quad g_1(x) = \frac{g(x)}{||g||} = \sqrt{\frac{5}{2}}\, g(x)$$

We obtain the set $S_1 = \{f_1, g_1\}$ which is orthonormal in $C_p^0[0,\ 2]$.

5. The trigonometric system

The following trigonometric identities will be used soon

$$\begin{aligned}
&2\cos A \cos B = \cos(A+B) + \cos(A-B)\\
&2\cos A \sin B = \sin(A+B) - \sin(A-B)\\
&2\sin A \sin B = \cos(A-B) - \cos(A+B)\\
&2\cos^2 A = \cos(2A) + 1\\
&2\sin^2 A = 1 - \cos(2A)
\end{aligned}$$

The trigonometric system over the interval $[0,\ 2\pi]$ (or over any interval of length 2π) consists of the functions

$$1,\ \cos x,\ \sin x,\ \cos(2x),\ \sin(2x), \cdots,\ \cos(kx),\ \sin(kx), \cdots, \qquad k \in \mathbb{Z}^+$$

Lemma. *The trigonometric system is orthogonal over* $[0,\ 2\pi]$.

Proof. We need to verify that the following inner products are zero.

$$\begin{aligned}
&< 1, \sin(kx) >=< 1, \cos(kx) >= 0 \qquad \forall k \in \mathbb{Z}^+\\
&< \cos(kx), \sin(lx) >= 0 \qquad \forall k, l \in \mathbb{Z}^+\\
&< \cos(kx), \cos(lx) >=< \sin(kx), \sin(lx) >= 0 \qquad \forall k, l \in \mathbb{Z}^+,\ k \neq l
\end{aligned}$$

We have for $k \in \mathbb{Z}^+$

$$< 1, \sin(kx) >= \int_0^{2\pi} \sin(kx)dx = \left(-\frac{\cos(kx)}{k}\right)_0^{2\pi} = -\frac{\cos(2k\pi) - \cos 0}{k} = 0$$

for $k, l \in \mathbb{Z}^+$ and $k \neq l$, we use one of the above trig identities to get

$$\begin{aligned}
< \cos(kx), \sin(lx) > &= \int_0^{2\pi} \cos(kx)\sin(lx)dx \\
&= \frac{1}{2}\int_0^{2\pi} (\sin(k+l)x - \sin(k-l)x)dx \\
&= \frac{-1}{2}\left(\frac{\cos(k+l)x}{k+l} - \frac{\cos(k-l)x}{k-l}\right)_0^{2\pi} \\
&= 0
\end{aligned}$$

When $k = l$, we have

$$< \cos(kx), \sin(kx) >= \frac{1}{2}\int_0^{2\pi} \sin(2kx)dx = \frac{-\cos(4k\pi) + \cos 0}{4k} = 0.$$

For $k, l \in \mathbb{Z}^+$ and $k \neq l$, we have

$$\begin{aligned}
< \sin(kx), \sin(lx) > &= \int_0^{2\pi} \sin(kx)\sin(lx)dx \\
&= \frac{1}{2}\int_0^{2\pi} (\cos(k+l)x - \cos(k-l)x)dx \\
&= \frac{1}{2}\left(\frac{\sin(k+l)x}{k+l} - \frac{\sin(k-l)x}{k-l}\right)_0^{2\pi} \\
&= 0
\end{aligned}$$

The verification of the remaining identities is left as an exercise.

Lemma. *The norm of the trig functions on* $[0,\ 2\pi]$ *are:*

$$||1|| = \sqrt{2\pi}, \quad ||\cos(kx)|| = ||\sin(kx)|| = \sqrt{\pi}.$$

Proof. We have

$$\begin{aligned}
||1||^2 &= \int_0^{2\pi} dx = 2\pi \\
||\cos(kx)||^2 &= \int_0^{2\pi} \cos^2(kx)dx = \int_0^{2\pi} \frac{1+\cos(2kx)}{2} \\
&= \left(\frac{x}{2} + \frac{\sin(2kx)}{4k}\right)_0^{2\pi} = \pi \\
||\sin(kx)||^2 &= \int_0^{2\pi} \sin^2(kx)dx = \int_0^{2\pi} \frac{1-\cos(2kx)}{2} \\
&= \left(\frac{x}{2} - \frac{\sin(2kx)}{4k}\right)_0^{2\pi} = \pi
\end{aligned}$$

As a consequence of the two lemmas we have

Corollary. *The following system is orthonormal over* $[0,\ 2\pi]$*:*

$$\frac{1}{\sqrt{2\pi}}, \ \frac{\sin x}{\sqrt{\pi}}, \ \frac{\cos x}{\sqrt{\pi}}, \ \cdots, \ \frac{\sin kx}{\sqrt{\pi}}, \ \frac{\cos kx}{\sqrt{\pi}}, \ \cdots$$

As we will see later the trigonometric system forms a basis for the space of piecewise continuous and 2π-periodic functions.

6. Fourier series of 2π-periodic functions

Let $f \in C^1_p(\mathbb{R})$ and 2π-periodic, we would like to associate to the function f a series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

in such a way that at each point x_0 where f is continuous the values of f and the series are the same:

$$f(x_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx_0) + b_n \sin(nx_0) \ .$$

An immediate question is the following. If a given function f has such a representation, how can we find the coefficients a_0, $a_1, b_1, \cdots$ in terms of f?

The answer is not really difficult if we assume that we can interchange the summation and the integration. Then by using the orthogonality of the trigonometric system, we would get

$$< f, 1 >= \int_0^{2\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx)dx + a_n \int_0^{2\pi} \sin(nx)dx = \pi a_0$$

Therefore,

$$a_0 = \frac{< f, 1 >}{\pi} = \frac{1}{\pi} \int_0^{2\pi} f(x)dx \ .$$

To find a_1 and b_1:

$$\begin{aligned} < f, \cos x > &= \frac{a_0}{2} < 1, \cos x > + \sum_{n=1}^{\infty} a_n < \cos(nx), \cos x > + b_n < \sin(nx), \cos x > \\ &= a_1 < \cos x, \cos x >= a_1 \, ||\cos x||^2 = a_1 \pi \\ < f, \sin x > &= \frac{a_0}{2} < 1, \sin x > + \sum_{n=1}^{\infty} a_n < \cos(nx), \sin x > + b_n < \sin(nx), \sin x > \\ &= b_1 < \sin x, \sin x >= b_1 \, ||\sin x||^2 = b_1 \pi \end{aligned}$$

Hence

$$\begin{aligned} a_1 &= \frac{< f, \cos x >}{\pi} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x dx \ ; \\ b_1 &= \frac{< f, \sin x >}{\pi} = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin x dx \ . \end{aligned}$$

Similar arguments give

$$\begin{aligned} a_k &= \frac{< f, \cos kx >}{||\cos kx||^2} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx \ ; \\ b_k &= \frac{< f, \sin kx >}{||\sin kx||^2} = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx \ . \end{aligned}$$

We have therefore established the association

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

for a 2π-periodic function $f \in C_p^0(\mathbb{R})$. The associated series is called the *Fourier series* of f. The coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\ a_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx \quad \text{for } k = 1, 2, \cdots \\ b_k &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx \quad \text{for } k = 1, 2, \cdots \end{aligned}$$

Remark 2. We have worked the integrations over the interval $[0,\ 2\pi]$. Because of the 2π-periodicity, we could have used any interval of length 2π. Hence, for any real number a, we also have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_a^{a+2\pi} f(x)dx \\ a_k &= \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos(kx)dx \quad \text{for } k = 1, 2, \cdots \\ b_k &= \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin(kx)dx \quad \text{for } k = 1, 2, \cdots \end{aligned}$$

In many situations with symmetry, it is useful to take $a = -\pi$. We have then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \quad \text{for } k = 1, 2, \cdots \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \quad \text{for } k = 1, 2, \cdots \end{aligned}$$

Remark 3. By using properties of even and odd functions, we have the following. If F is even, then the functions $F(x)\cos(kx)$ are even and the functions $F(x)\sin(kx)$ are odd. If G is odd, then the functions $G(x)\cos(kx)$ are odd and the functions $G(x)\sin(kx)$ are even.

The Fourier coefficients of an even function F are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)dx = \frac{2}{\pi}\int_0^{\pi} F(x)dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos(kx)dx = \frac{2}{\pi}\int_0^{\pi} F(x)\cos(kx)dx \quad \text{for } k = 1, 2, \cdots \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin(kx)dx = 0 \quad \text{for } k = 1, 2, \cdots \end{aligned}$$

The Fourier coefficients of an odd function G are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} G(x)dx = 0 \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} G(x)\cos(kx)dx = 0 \quad \text{for } k = 1, 2, \cdots \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} G(x)\sin(kx)dx = \frac{2}{\pi}\int_0^{\pi} G(x)\sin(kx) \quad \text{for } k = 1, 2, \cdots \end{aligned}$$

The Fourier series of an even function F and an odd function G are:

$$F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \; , \quad a_n = \frac{2}{\pi}\int_0^{\pi} F(x)\cos(nx)dx \; ;$$
$$G(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \; , \qquad b_n = \frac{2}{\pi}\int_0^{\pi} G(x)\sin(nx)dx \; .$$

7. Examples

We give examples of Fourier series of some simple functions

Example 1. Let $f(x)$ be the 2π-periodic function defined on $[-\pi,\ \pi]$ by $f(x) = |x|$. Hence the graph of f is the triangular wave. Note that f is an even function and

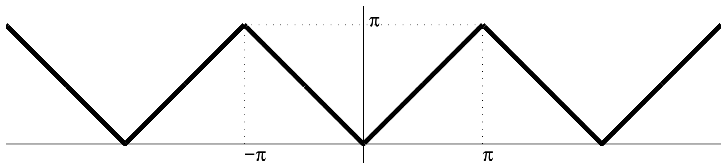

Figure 9. *Triangular wave*

so its Fourier series contains only the cosine terms ($b_n = 0$ for every $n \in \mathbb{Z}^+$). The coefficients of the cosines are:

$$\begin{aligned}
\frac{\pi}{2}a_0 &= \int_0^{\pi} x dx = \frac{\pi^2}{2} \; ; \\
\frac{\pi}{2}a_n &= \int_0^{\pi} x\cos(nx)dx = \left(x\frac{\sin(nx)}{n} \right)_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}dx \\
&= \left(\frac{\cos(nx)}{n^2} \right)_0^{\pi} = \frac{\cos(n\pi) - 1}{n^2} \\
&= \frac{(-1)^n - 1}{n^2}
\end{aligned}$$

Hence,

$$a_0 = \pi \; , \quad \text{and} \quad a_n = \frac{2((-1)^n - 1)}{\pi n^2} = \begin{cases} 0 & \text{if } n = 2k \text{ (even)} \\ \dfrac{-4}{\pi(2k+1)^2} & \text{if } n = 2k+1 \text{ (odd)} \end{cases}$$

We have obtained the associated Fourier series of f on $[0,\ 2\pi]$

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2} \; .$$

Example 2. Let $f(x)$ be the 2π-periodic function defined on $[-\pi,\ \pi]$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \; ; \\ -1 & \text{if } -\pi < x < 0 \; . \end{cases}$$

Hence the graph of f is the rectangular wave. Note that f is an odd function and

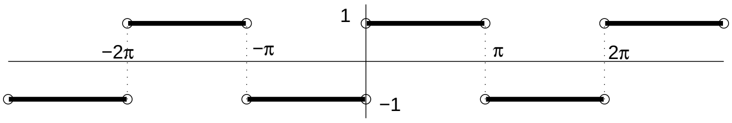

Figure 10. *Rectangular wave*

so its Fourier series contains only the sine terms ($a_0 = 0$ and $a_n = 0$ for every $n \in \mathbb{Z}^+$). The coefficients of the sines are:

$$\frac{\pi}{2} b_n = \int_0^\pi \sin(nx)dx = \left(\frac{-\cos(nx)}{n}\right)_0^\pi = \frac{1-(-1)^n}{n}$$

Hence,

$$b_n = \frac{2(1-(-1)^n)}{\pi n} = \begin{cases} 0 & \text{if } n = 2k \text{ (even)} \\ \dfrac{4}{\pi(2k+1)} & \text{if } n = 2k+1 \text{ (odd)} \end{cases}$$

We have obtained the associated Fourier series of f on $[0,\ 2\pi]$

$$f(x) \sim \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)} .$$

Example 3. Let $f(x)$ be the 2π-periodic function defined on $[-\pi,\ \pi]$ by

$$f(x) = \begin{cases} x & \text{if } 0 < x < \pi \ ; \\ 0 & \text{if } -\pi < x < 0 \ . \end{cases}$$

The graph of f is the given below. The function f is neither even nor odd and so

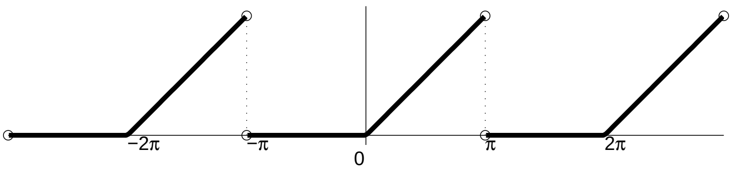

Figure 11. *Graph of f*

its Fourier series contains sines and cosines terms. Its Fourier coefficients are:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^\pi x dx = \frac{\pi}{2} \ ; \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_0^\pi x\cos(nx)dx = \frac{(-1)^n - 1}{\pi n^2} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_0^\pi x\sin(nx)dx = \frac{(-1)^{n-1}}{n} . \end{aligned}$$

We have

$$f(x) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\sin jx}{j} .$$

8. The complex exponential form of Fourier series

Recall that the cosine and sine function can be expressed in terms of the exponential as

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \qquad \begin{cases} \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = -i\dfrac{e^{i\theta}-e^{-i\theta}}{2} \end{cases}$$

Now we can rewrite the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \ ,$$

with real coefficients in terms of these exponentials. For this, we use

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= a_n \frac{e^{inx}+e^{-inx}}{2} - ib_n \frac{e^{inx}-e^{-inx}}{2} \\ &= \frac{a_n - ib_n}{2} e^{inx} + \frac{a_n + ib_n}{2} e^{-inx} \\ &= c_n e^{inx} + \overline{c_n} e^{-inx} \end{aligned}$$

where $\overline{c_n}$ denotes the complex conjugate of c_n

For a 2π-periodic, $\mathbb{R}$-valued function function $f \in C_p^0(\mathbb{R})$, we have

$$f(x) \ \sim \ c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \overline{c_n e^{inx}}$$

with

$$\begin{aligned} c_n &= \frac{a_n - ib_n}{2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \left(\cos nx - i \sin nx\right) dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \end{aligned}$$

This complex form of the Fourier is equivalent to the one given with cosines and sines. In many applications it is easier and more convenient to use the complex form.

Note that the above association take the form

$$f(x) \ \sim \ c_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}\left(c_n e^{inx}\right)$$

where $\mathrm{Re}(C)$ denotes the real part of the complex number C.

Example 1. Let f be the 2π-periodic function defined over the interval $(-\pi,\ \pi)$ by

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x^2 & \text{if } 0 < x < \pi \end{cases}$$

To compute the Fourier coefficients, we use integration by parts.

$$\begin{aligned}\int x^2 \mathrm{e}^{-inx} dx &= \frac{x^2\mathrm{e}^{-inx}}{-in} - \int \frac{2x\mathrm{e}^{-inx}}{-in} dx \\ &= \frac{x^2\mathrm{e}^{-inx}}{-in} - \frac{2x\mathrm{e}^{-inx}}{(-in)^2} + \int \frac{2\mathrm{e}^{-inx}}{(-in)^2} dx \\ &= \frac{x^2\mathrm{e}^{-inx}}{-in} - \frac{2x\mathrm{e}^{-inx}}{(-in)^2} + \frac{2\mathrm{e}^{-inx}}{(-in)^3} + C \\ &= i\frac{x^2\mathrm{e}^{-inx}}{n} + \frac{2x\mathrm{e}^{-inx}}{n^2} - i\frac{2\mathrm{e}^{-inx}}{n^3} + C\end{aligned}$$

We find then the Fourier coefficients

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} = \frac{1}{2\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{6}$$

and for $n \geq 1$,

$$\begin{aligned}c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx} dx = \frac{1}{2\pi}\int_0^{\pi} x^2\mathrm{e}^{-inx} dx \\ &= \frac{1}{2\pi}\left(i\frac{x^2\mathrm{e}^{-inx}}{n} + \frac{2x\mathrm{e}^{-inx}}{n^2} - i\frac{2\mathrm{e}^{-inx}}{n^3}\right)_0^{\pi}\end{aligned}$$

It gives

$$c_n = \frac{(-1)^n}{n^2} + i\left(\frac{\pi(-1)^n}{2n} + \frac{1-(-1)^n}{\pi n^3}\right)$$

The Fourier series of f is therefore

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \operatorname{Re}\left[c_n \mathrm{e}^{inx}\right]$$

We can rewrite the Fourier series in terms of cosine and sine. First

$$\begin{aligned}c_n\mathrm{e}^{inx} &= \frac{(-1)^n}{n^2}\cos nx - \left(\frac{\pi(-1)^n}{2n} + \frac{1-(-1)^n}{\pi n^3}\right)\sin nx + \\ &+ i\left(\frac{(-1)^n}{n^2}\sin nx - \left(\frac{\pi(-1)^n}{2n} + \frac{1-(-1)^n}{\pi n^3}\right)\cos nx\right)\end{aligned}$$

The Fourier series of f is:

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - \left(\frac{\pi(-1)^n}{2n} + \frac{1-(-1)^n}{\pi n^3}\right)\sin nx$$

Example 2. Let f be the 2π-periodic function defined on $[0,\ 2\pi]$ as $f(x) = \mathrm{e}^x$. The n-th Fourier coefficient of f is

$$\begin{aligned}c_n &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^x\mathrm{e}^{-inx} dx = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{(1-in)x} dx \\ &= \frac{1}{2\pi}\left(\frac{\mathrm{e}^{(1-in)x}}{1-in}\right)_{x=0}^{x=2\pi} \\ &= \frac{(\mathrm{e}^{2\pi}-1)}{2\pi}\frac{1}{1-in} = \frac{(\mathrm{e}^{2\pi}-1)}{2\pi}\frac{1+in}{1+n^2}\end{aligned}$$

Hence (after a calculation), we find

$$\operatorname{Re}(c_n\mathrm{e}^{inx}) = \frac{(\mathrm{e}^{2\pi}-1)}{2\pi}\left[\frac{\cos nx}{1+n^2} - \frac{n\sin nx}{1+n^2}\right]$$

The Fourier series of f is therefore,

$$\begin{aligned} \mathrm{e}^x &\sim c_0 + 2\sum_{n=1}^{\infty} \mathrm{Re}(c_n \mathrm{e}^{inx}) \\ &\sim \frac{(\mathrm{e}^{2\pi}-1)}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} \frac{\cos nx}{1+n^2} - \frac{n \sin nx}{1+n^2}\right] \end{aligned}$$

9. Exercises

In each exercise, find the fourier series of the 2π-periodic function f that is given by

Exercise 1. $f(x) = x$ for $-\pi < x < \pi$.

Exercise 2. $f(x) = \pi - x$ for $0 < x < 2\pi$.

Exercise 3. $f(x) = \cos^2 x$

Exercise 4. $f(x) = \begin{cases} -x & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$

Exercise 5. $f(x) = |\cos x|$

Exercise 6. $f(x) = |\sin x|$

Exercise 7. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 < x < \pi \end{cases}$

Exercise 8. $f(x) = \begin{cases} 1/(2d) & \text{if } |x| < d \\ 0 & \text{if } d < |x| < \pi \end{cases}$ with d a positive constant.

Exercise 9. $f(x) = \mathrm{e}^{dx}$ for $-\pi < x < \pi$, where d a positive constant

Exercise 10. $f(x) = \cosh x$ for $-\pi < x < \pi$.

Exercise 11. $f(x) = \cosh x$ for $0 < x < 2\pi$.

Exercise 12. $f(x) = \cosh x$ for $0 < x < 2\pi$.

Exercise 13. $f(x) = \begin{cases} (a - |x|)/(2d) & \text{if } |x| < d \\ 0 & \text{if } d < |x| < \pi \end{cases}$ with d a positive constant.