FOURIER SERIES PART II: CONVERGENCE

We have seen in the previous note how to associate to a 2π-periodic function f a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \ .$$

Now we are going to investigate how the Fourier series represents f. Let us first introduce the following notation. For $N = 0,\ 1,\ 2,\ \cdots$, we denote by $S_N f(x)$ the N-th partial sum of the Fourier series of f. That is,

$$S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx) \ .$$

Hence

$$\begin{aligned} S_0 f(x) &= \frac{a_0}{2} \ ; \\ S_1 f(x) &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x \ ; \\ S_2 f(x) &= \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \\ &\vdots \end{aligned}$$

The infinite series is therefore $\lim_{N\to\infty} S_N f$. The Fourier series *converges* at a point x if $\lim_{N\to\infty} S_N f(x)$ exists.

We consider the functions and their Fourier series of examples 1, 2, and 3 of the previous note and see how the graphs of partial sums $S_N f$ compare to those of f.

1. Examples

Example 1. For $f(x) = |x|$ on $[-\pi,\ \pi]$, we found

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2}$$

Thus,

$$\begin{aligned} S_0 f(x) &= \frac{\pi}{2} \ ; \\ S_1 f(x) &= \frac{\pi}{2} - \frac{4\cos x}{\pi} \ ; \\ S_3 f(x) &= \frac{\pi}{2} - \frac{4\cos x}{\pi} - \frac{4\cos 3x}{9\pi} \ ; \\ S_5 f(x) &= \frac{\pi}{2} - \frac{4\cos x}{\pi} - \frac{4\cos 3x}{9\pi} - \frac{4\cos 5x}{25\pi} \end{aligned}$$

It appears that as N gets larger, the graph of $S_N f$ gets closer to that of f.

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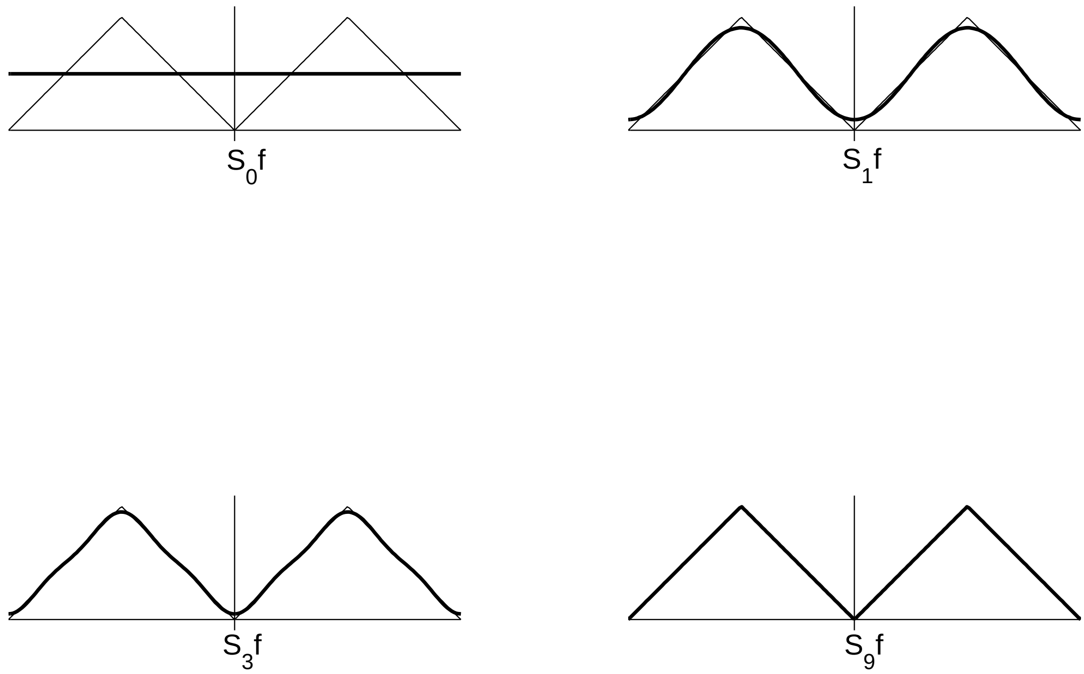

FIGURE 1. *Graphs of $S_N f$ for $N = 0,\ 1,\ 3,\ 9$.*

Example 2. For the 2π-periodic function f of example 2 defined by $f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \ ; \\ -1 & \text{if } -\pi < x < 0 \end{cases}$, we found the Fourier series

$$f(x) \sim \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)} \ .$$

Thus,

$$\begin{aligned} S_1 f(x) &= \frac{4 \sin x}{\pi} \ ; \\ S_3 f(x) &= \frac{4 \sin x}{\pi} + \frac{4 \sin 3x}{3\pi} \ ; \\ S_5 f(x) &= \frac{4 \sin x}{\pi} + \frac{4 \sin 3x}{3\pi} + \frac{4 \sin 5x}{5\pi} \\ S_7 f(x) &= \frac{4 \sin x}{\pi} + \frac{4 \sin 3x}{3\pi} + \frac{4 \sin 5x}{5\pi} + \frac{4 \sin 7x}{7\pi} \end{aligned}$$

Again it appears that as N increases $S_N f$ gets closer to f at the points where f is continuous.

Example 3. For the 2π-periodic function f of example 3 defined by $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \ ; \\ 0 & \text{if } -\pi < x < 0 \ . \end{cases}$ with Fourier series

$$\frac{\pi}{4} - \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \sin jx}{j} \ .$$

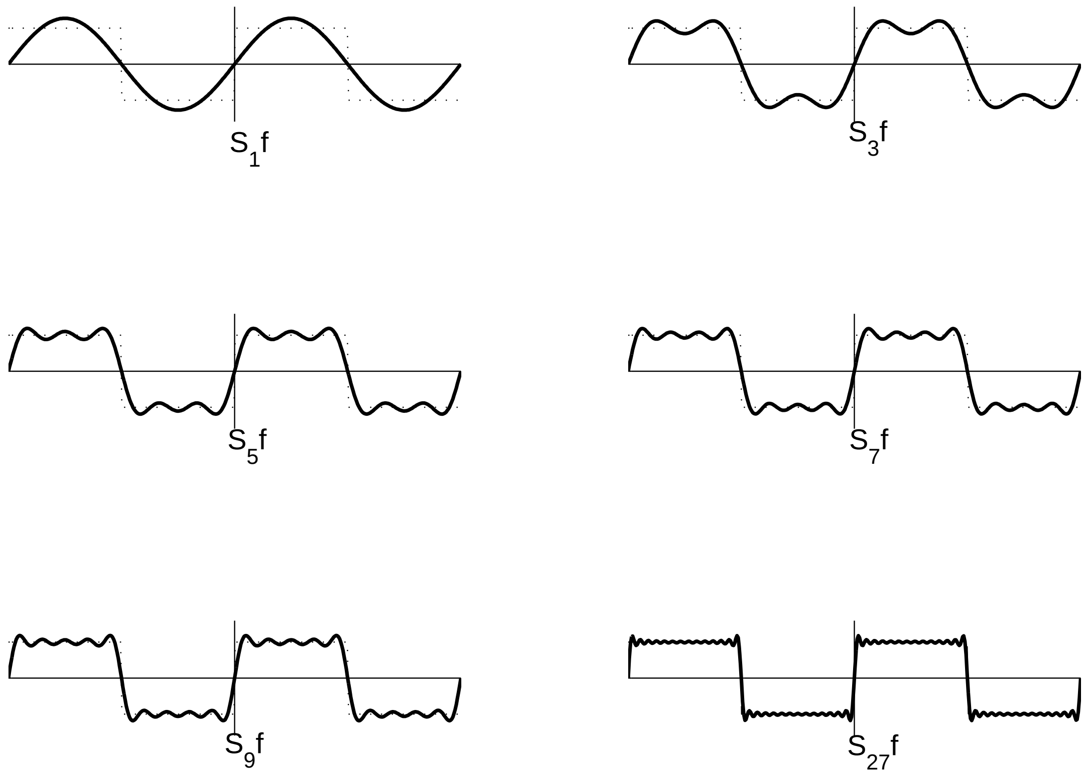

FIGURE 2. *Graphs of* $S_N f$ *for* $N = 1,\ 3,\ 5,\ 7,\ 9,\ 27$.

The first partial sums are

$$\begin{aligned} S_0 f(x) &= \frac{\pi}{4} \\ S_1 f(x) &= \frac{\pi}{4} - \frac{2\cos x}{\pi} + \sin x \ ; \\ S_2 f(x) &= \frac{\pi}{4} - \frac{2\cos x}{\pi} + \sin x - \frac{\sin 2x}{2} \ ; \\ S_3 f(x) &= \frac{\pi}{4} - \frac{2\cos x}{\pi} + \sin x - \frac{\sin 2x}{2} - \frac{2\cos 3x}{9\pi} + \frac{\sin 3x}{3} \ . \end{aligned}$$

2. Pointwise Convergence of Fourier series

The above examples suggest that the N-th partial sums $S_N f$ converge to f. This is indeed the case at each point where f is continuous. At each discontinuity, the partial sums approach the average value of f. To be precise, we define the average of f at a point x_0 as

$$f_{av}(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2} = \frac{1}{2}\left(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\right) \ .$$

Hence if f is continuous at x_0, then $f_{av}(x_0) = f(x_0)$. For example for the 2π-periodic function f of example 3 defined by $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \ ; \\ 0 & \text{if } -\pi < x < 0 \ . \end{cases}$ we

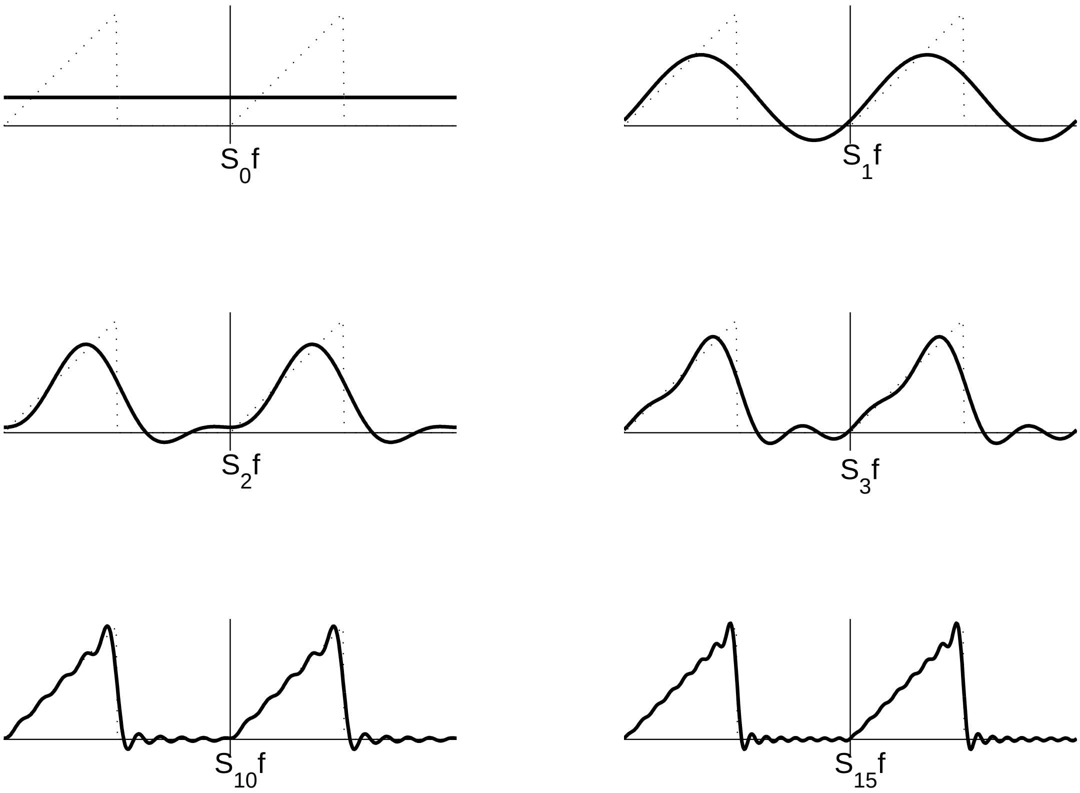

FIGURE 3. *Graphs of $S_N f$ for $N = 0,\ 1,\ 2,\ 3,\ 10,\ 15$.*

have $f_{av}(x) = f(x)$ for $x \neq (2k+1)\pi$ (with $k \in \mathbb{Z}$) and

$$f_{av}((2k+1)\pi) = \frac{f((2k+1)\pi^+) + f((2k+1)\pi^-)}{2} = \frac{\pi}{2} \quad k = \pm 1,\ \pm 2,\ \pm 3,\ \cdots$$

The graph of f_{av} is the following

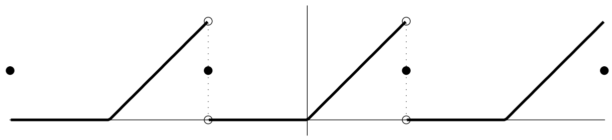

FIGURE 4. *Graphs of f_{av}.*

We have the following theorem.

Theorem (Pointwise convergence) *Let $f \in C^1_p(\mathbb{R})$ be 2π-periodic. Then the Fourier series of f converges to f_{av} at each point of $\mathbb{R}$. That is,*

$$f_{av}(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx\ ,$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \ , \ \text{and} \quad \left.\begin{array}{c} a_n \\ b_n \end{array}\right\} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \left\{\begin{array}{c} \cos nx \\ \sin nx \end{array}\right. dx$$

Again this means that at all points x where f is continuous, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx \ ,$$

and at the points x_0 where f is discontinuous we have

$$\frac{f(x_0^+) + f(x_0^-)}{2} = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx_0 + b_n \sin nx_0 \ .$$

To prove this theorem, we will need two lemmas

Lemma 1. (Riemann-Lebesgue Lemma) *If f is piecewise smooth on an interval $[a,\ b]$, then*

$$\lim_{r\to\infty}\int_a^b f(x)\cos(rx)dx = 0$$
$$\lim_{r\to\infty}\int_a^b f(x)\sin(rx)dx = 0$$

Proof. Since f is piecewise smooth, then there are finitely many points

$$c_0 = a \ < \ c_1 \ < \ c_2 \ < \ \cdots \ < \ c_{n-1} \ < \ b = c_n$$

such that both f and its derivative f' are continuous in each interval $(c_{j-1},\ c_j)$ $(j = 1, \cdots, n)$. Furthermore, $f(c_k^{\pm})$ and $f'(c_k^{\pm})$ are finite. Thus the integrals of f and f' exist in each subinterval. We have,

$$\begin{aligned}\int_a^b f(x)\cos(rx)dx &= \int_{c_0}^{c_1} f(x)\cos(rx)dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)\cos(rx)dx \\ &= \sum_{j=1}^{n}\int_{c_{j-1}}^{c_j} f(x)\cos(rx)dx\end{aligned}$$

We use integration by parts in each subinterval $[c_{j-1},\ c_j]$ to obtain

$$\int_{c_{j-1}}^{c_j} f(x)\cos(rx)dx = \left(\frac{f(x)\sin(rx)}{r}\right)_{c_{j-1}}^{c_j} - \int_{c_{j-1}}^{c_j} f'(x)\frac{\sin(rx)}{r}dx$$

(we are assuming that $r > 0$). Let $M > 0$ such that

$$\sup_{a<x<b} |f(x)| < M \quad \text{and} \quad \sup_{a<x<b} |f'(x)| < M \, .$$

Then

$$\left|\left(\frac{f(x)\sin(rx)}{r}\right)_{c_{j-1}}^{c_j}\right| \leq \left|\frac{f(c_j)\sin(rc_j)}{r}\right| + \left|\frac{f(c_{j-1})\sin(rc_{j-1})}{r}\right| \leq \frac{2M}{r}$$

and

$$\left|\int_{c_{j-1}}^{c_j} f'(x)\frac{\sin(rx)}{r}dx\right| \leq \int_{c_{j-1}}^{c_j}\left|f'(x)\frac{\sin(rx)}{r}\right| dx \leq \frac{M(c_j - c_{j-1})}{r}.$$

It follows that

$$\left|\int_a^b f(x)\cos(rx)dx\right| \le \sum_{j=1}^{n}\left(\frac{2M}{r}+\frac{M(c_j-c_{j-1})}{r}\right) \le \frac{2Mn+(b-a)}{r}.$$

Since $\dfrac{2Mn+(b-a)}{r} \to 0$ as $r\to\infty$, then

$$\lim_{r\to\infty}\int_a^b f(x)\cos(rx)dx = 0.$$

A similar argument gives the second limit of the lemma.

Lemma 2. *for every $x\in\mathbb{R}$, $x\neq 2k\pi$ with $k\in\mathbb{Z}$, we have the identity*

$$\frac{1}{2}+\cos x+\cos(2x)+\cdots+\cos(Nx) = \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}}\ .$$

Proof. Set $T=\dfrac{1}{2}+\cos x+\cos(2x)+\cdots+\cos(Nx)$. By using

$$\cos\theta = \frac{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}}{2}\ ,$$

we can rewrite T as

$$T=\frac{1}{2}+\sum_{j=1}^{N}\frac{\mathrm{e}^{ijx}+\mathrm{e}^{-ijx}}{2} = \frac{1}{2}\left(1+\sum_{j=1}^{N}\mathrm{e}^{ijx}+\sum_{j=1}^{N}\mathrm{e}^{-ijx}\right)$$

Note that $\sum_{j=1}^{N}\mathrm{e}^{ijx}$ and $\sum_{j=1}^{N}\mathrm{e}^{-ijx}$ are geometric sums. The first with ratio e^{ix} and the second with ratio e^{-ix}. Since $x\neq 2k\pi$ these ratios are different from 1 and

$$\sum_{j=1}^{N}\mathrm{e}^{ijx} = \frac{\mathrm{e}^{ix}(1-\mathrm{e}^{iNx})}{1-\mathrm{e}^{ix}} = \frac{\mathrm{e}^{ix}-\mathrm{e}^{i(N+1)x}}{1-\mathrm{e}^{ix}}$$

and

$$\sum_{j=1}^{N}\mathrm{e}^{-ijx} = \frac{\mathrm{e}^{-ix}(1-\mathrm{e}^{-iNx})}{1-\mathrm{e}^{-ix}} = \frac{\mathrm{e}^{-ix}-\mathrm{e}^{-i(N+1)x}}{1-\mathrm{e}^{-ix}}\ .$$

After reducing to the same denominator, the expression for T becomes

$$T=\frac{\mathrm{e}^{iNx}+\mathrm{e}^{-iNx}-\mathrm{e}^{i(N+1)x}-\mathrm{e}^{-i(N+1)x}}{2(2-(\mathrm{e}^{ix}+\mathrm{e}^{-ix}))} = \frac{\cos Nx-\cos(N+1)x}{2(1-\cos x)}\ .$$

Now use the trigonometric identities

$$\cos Nx = \cos((N+\tfrac{1}{2})x-\tfrac{x}{2}) = \cos((N+\tfrac{1}{2})x)\cos\frac{x}{2}+\sin((N+\tfrac{1}{2})x)\sin\frac{x}{2}$$
$$\cos(N+1)x = \cos((N+\tfrac{1}{2})x+\tfrac{x}{2}) = \cos((N+\tfrac{1}{2})x)\cos\frac{x}{2}-\sin((N+\tfrac{1}{2})x)\sin\frac{x}{2}\ .$$

Hence,

$$\cos Nx-\cos(N+1)x = 2\sin((N+\tfrac{1}{2})x)\sin\frac{x}{2}\ .$$

We also have

$$1-\cos x = 2\sin^2\frac{x}{2}\ .$$

Therefore

$$T = \frac{2\sin((N+\dfrac{1}{2})x)\sin\dfrac{x}{2}}{4\sin^2\dfrac{x}{2}} = \frac{\sin(N+\frac{1}{2})x}{2\sin\dfrac{x}{2}} \; .$$

After these two lemmas, we start the proof of the convergence of Fourier series. Let $S_N f$ be the N-th partial sum of the Fourier series of f. That is,

$$S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx) \; .$$

We would like to prove that

$$\lim_{N\to\infty} S_N f(x) = f_{av}(x) = \frac{f(x^+) + f(x^-)}{2} \; .$$

We are going to use Lemma 2 and the definition of the Fourier coefficients a_j and b_j to rewrite $S_N f$ in an integral form. Recall that

$$\begin{aligned} a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt dt\,, \quad j = 0,\ 1,\ 2,\ \cdots \\ b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt dt\,, \quad j = 1,\ 2,\ 3,\ \cdots \end{aligned}$$

We can rewrite $S_N f$ as

$$\begin{aligned} S_N f(x) \quad &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos jt dt + \\ &+ \sum_{j=1}^{N}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos jt dt \cos jx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin jt dt \sin jx\right) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{j=1}^{N}(\cos jt\cos jx + \sin jt \sin jx)\right) dt \end{aligned}$$

The trigonometric identity $\cos jt\cos jx + \sin jt\sin jx = \cos j(t-x)$ gives

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{j=1}^{N}\cos j(t-x)\right) dt$$

Now Lemma 2 can be used to obtain

$$S_N(x) = \int_{-\pi}^{\pi} f(t)\frac{\sin\left[(N+\dfrac{1}{2})(t-x)\right]}{2\pi\sin\left[\dfrac{t-x}{2}\right]} dt$$

Define the function $D_N(s)$, called the *Dirichlet kernel*, by

$$D_N(s) = \begin{cases} \dfrac{\sin(N+\dfrac{1}{2})s}{2\pi\sin\dfrac{s}{2}} & \text{if } s \neq 2k\pi\,,\ k\in\mathbb{Z}\,; \\ \dfrac{2N+1}{2\pi} & \text{if } s = 2k\pi\ \ k\in\mathbb{Z}\,. \end{cases}$$

Note that $D_N : \mathbb{R} \longrightarrow \mathbb{R}$ is an even and continuous function. That D_N is contin-

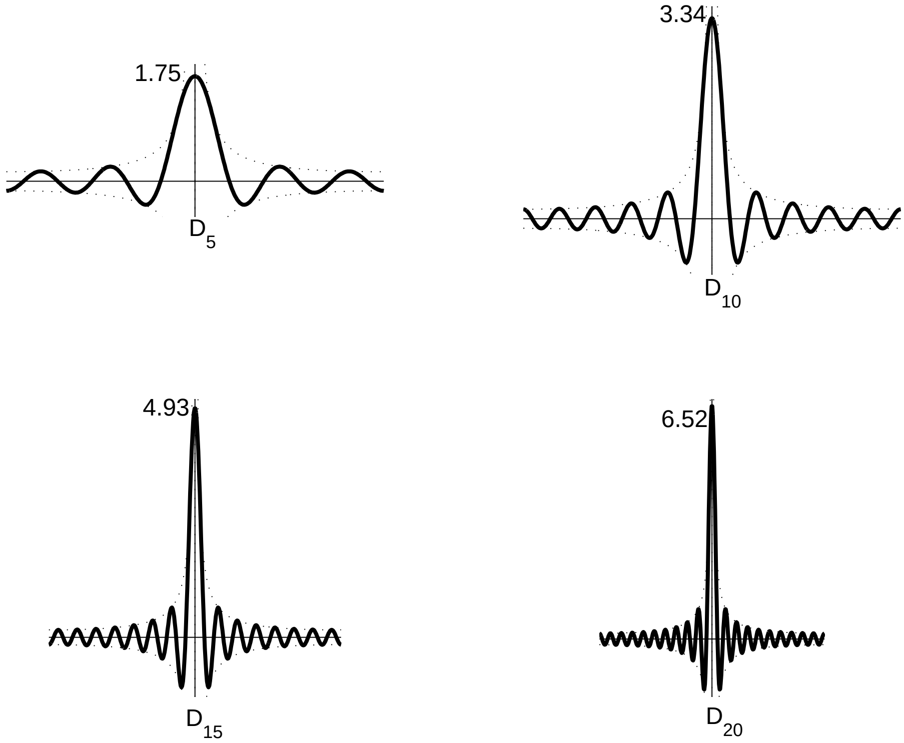

FIGURE 5. *Graphs of D_5, D_{10}, D_{15} and D_{20} over the interval* $[-\pi,\ \pi]$

uous at a point $s_0 = 2k\pi$ follows from L'Hopital's rule:

$$\lim_{s\to 2k\pi} D_N(s) = \lim_{s\to 2k\pi} \frac{(N+\frac{1}{2})\cos(N+\frac{1}{2})s}{\pi\cos\frac{s}{2}} = \frac{2N+1}{2\pi} = D_N(2k\pi)\ .$$

Furthermore, D_N is 2π-periodic. We will use the integral of D_N. We have from Lemma 2 that

$$\int_0^{\pi} D_N(s)ds = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{2}+\cos s+\cos(2s)+\cdots+\cos(Ns)\right)ds = \frac{1}{2}$$

(since $\int_0^{\pi} cosjxdx = 0$ for $j = 1,\ 2,\ 3,\cdots$). We also have

$$\int_{-\pi}^{0} D_N(s)ds = \frac{1}{2}$$

since the function D_N is even.

So far we proved that

$$S_N f(x) = \int_{-\pi}^{\pi} f(t)D_N(t-x)dt\ .$$

By using the substitution $s = t - x$, and by using the 2π-periodicity of f and of D_N, we rewrite $S_N f$ as

$$S_N f(x) = \int_{-\pi-x}^{\pi-x} f(x+s)D_N(s)ds = \int_{-\pi}^{\pi} f(x+s)D_N(s)ds \ .$$

Since

$$S_N f(x) = \int_{-\pi}^{0} f(x+s)D_N(s)ds \ + \int_{0}^{\pi} f(x+s)D_N(s)ds \ ,$$

then to prove that $\lim_{N\to\infty} S_N f(x) = f_{av}(x)$, it is enough to prove that

$$\lim_{N\to\infty}\int_{0}^{\pi} f(x+s)D_N(s)ds = \frac{f(x^+)}{2} \quad \text{and}$$
$$\lim_{N\to\infty}\int_{-\pi}^{0} f(x+s)D_N(s)ds = \frac{f(x^-)}{2} \ .$$

For this, we consider the functions $h(s)$ and $k(s)$ defined in $[-\pi,\ \pi]$ by

$$h(s) = \begin{cases} \dfrac{f(x+s)-f(x^-)}{s} & \text{for } s<0 \\ \dfrac{f(x+s)-f(x^+)}{s} & \text{for } s>0 \end{cases} \quad \text{and} \quad k(s) = \begin{cases} \dfrac{s}{2\sin(s/2)} & \text{for } s\neq 0 \\ 1 & \text{for } s = 0 \end{cases}$$

I leave it as an exercise for you to verify that k and its derivative are continuous on $[-\pi,\ \pi]$ (use L'Hopital's rule at $s = 0$). For the function h, it is piecewise smooth in each closed interval not containing $s = 0$. At $s = 0$, we have

$$h(0^-) = \lim_{s\to 0^-} \frac{f(x+s)-f(x^-)}{s} = f'(x^-)$$
$$h(0^+) = \lim_{s\to 0^+} \frac{f(x+s)-f(x^+)}{s} = f'(x^+)$$

Hence h is piecewise continuous on $[-\pi,\ \pi]$.

Claim. *We have the following*

$$\lim_{N\to\infty}\int_{0}^{\pi} h(s)k(s)\sin(N+\frac{1}{2})s\,ds = 0$$
$$\lim_{N\to\infty}\int_{-\pi}^{0} h(s)k(s)\sin(N+\frac{1}{2})s\,ds = 0$$

Proof of claim. We prove the first limit. Let $\epsilon > 0$. Since, the integrand is piecewise continuous and uniformly bounded, we can find $\delta > 0$ such that

$$\left|\int_{0}^{\delta} h(s)k(s)\sin(N+\frac{1}{2})s\,ds\right| < \epsilon \qquad \forall N \in \mathbb{Z}^+.$$

The integrand is piecewise smooth on the interval $[\delta,\ \pi]$. Lemma 1 implies that

$$\lim_{N\to\infty}\int_{\delta}^{\pi} h(s)k(s)\sin(N+\frac{1}{2})s\,ds = 0.$$

Hence,

$$\lim_{N\to\infty}\int_{0}^{\pi} h(s)k(s)\sin(N+\frac{1}{2})s\,ds \leq \epsilon$$

Since $\epsilon > 0$ is arbitrary, then the limit is 0. The second limit of the claim is proved in a similar way and is left as an exercise.

End of the proof of the theorem. By using $\int_0^\pi D_N(s)ds = \frac{1}{2}$, we get

$$\begin{aligned}\int_0^\pi f(x+s)D_N(s)ds - \frac{f(x^+)}{2} &= \int_0^\pi (f(x+s) - f(x^+))D_N(s)ds \\ &= \frac{1}{\pi}\int_0^\pi \frac{f(x+s)-f(x^+)}{s}\frac{s}{2\sin(s/2)}\sin(N+\frac{1}{2})sds \\ &= \frac{1}{\pi}\int_0^\pi h(s)k(s)\sin(N+\frac{1}{2})sds\end{aligned}$$

It follows from this and the claim that

$$\lim_{N\to\infty}\int_0^\pi f(s+u)D_N(s)ds = \frac{f(x^+)}{2} .$$

A similar argument gives

$$\lim_{N\to\infty}\int_{-\pi}^0 f(s+u)D_N(s)ds = \frac{f(x^-)}{2}$$

and completes the proof of the theorem.

Example 1. The 2π-periodic function f defined on $[-\pi,\ \pi]$ by $f(x) = |x|$ (the triangular wave function) is continuous on $\mathbb{R}$. It is therefore equal to its Fourier series for all $x \in \mathbb{R}$. In particular,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2}\ , \quad \forall x \in [-\pi,\ \pi]\ .$$

Fourier can be used to evaluate numerical series. For $x = 0$, we obtain

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)0}{(2j+1)^2}$$

Hence,

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Example 2. The 2π-periodic function f defined on $[-\pi,\ \pi]$ by $f(x) = 1$ if $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$ is continuous everywhere except at the points $k\pi$, with $k \in \mathbb{Z}$. Thus $f(x)$ equal its Fourier series for $x \neq k\pi$. In particular

$$1 = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)}\ , \quad \forall x \in (0,\ \pi)\ .$$

For $x = k\pi$, we have $f_{av}(k\pi) = 0$ which is the value of the Fourier series when $x = k\pi$. We can use this series to evaluate the alternating series $\sum_{j=0}^{\infty}(-1)^j/(2j+1)$. Indeed, for $x = \frac{\pi}{2}$, we get

$$1 = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)\pi/2}{(2j+1)} = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)}\ .$$

Hence,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}\ .$$

Example 3. The 2π-periodic function f defined on $[-\pi,\ \pi]$ by $f(x) = x$ if $0 \leq x < \pi$ and $f(x) = 0$ for $-\pi < x \leq 0$ is continuous everywhere except at the points $(2k+1)\pi$, with $k \in \mathbb{Z}$. The Fourier series of f is therefore equal to $f(x)$ everywhere except at the points $(2k+1)\pi$. We have then,

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2} + \sum_{j=1}^{\infty}\frac{(-1)^{j-1}\sin jx}{j} = \begin{cases} x & \text{if } 0 \leq x < \pi \\ 0 & \text{if } -\pi < x \leq 0 \end{cases} .$$

At the points $(2k+1)\pi$, we have the average value $f_{av}((2k+1)\pi) = \pi/2$. At such points the Fourier series is $\pi/2$.

3. Differentiation of Fourier Series

When dealing with series of functions, one has to be careful on whow to use termwise differentiation. Consider the function of example 2 defined by $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$. We found the Fourier series of f. We have in particular

$$1 = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)}\ , \quad \forall x \in (0,\ \pi)\ .$$

We are tempted to differentiate and write

$$0 = \frac{4}{\pi}\sum_{j=0}^{\infty}\cos(2j+1)x\ .$$

But this cannot be the case since the series diverges. To be able to use term by term differentiation we need an extra condition on f. More precisely, we have

Theorem. *Let f be 2π-periodic and* **continuous** *function on $\mathbb{R}$ such that its derivative f' is piecewise smooth. Let $a_0,\ a_1,\ b_1,\ a_2,\ b_2,\ \cdots,$ be the Fourier coefficients of f. Then the Fourier coefficients $a_0',\ a_1',\ b_1',\ a_2',\ b_2',\ \cdots$ of f' are*

$$a_0' = 0, \qquad a_n' = nb_n, \quad b_n' = -na_n$$

Remark. Under the hypotheses of the theorem, we have term by term differentiation. So if f is continuous, 2π-periodic and f' is piecewise smooth, then

$$\begin{aligned} f(x) \quad &= \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx \\ f'(x) \quad &\sim \sum_{n=0}^{\infty} nb_n \cos nx - na_n \sin nx \end{aligned}$$

Claim. *If g is continuous on an interval $[a,\ b]$ and g' is piecewise continuous on $[a,\ b]$, then*

$$\int_a^b g'(x)dx = g(b) - g(a)\ .$$

Proof of the claim. Let $c_0 = a < c_1 < c_2 < \cdots < c_n = b$ be the possible jump discontinuities of g'. Then,

$$\begin{aligned}
\int_a^b g'(x)dx &= \sum_{j=1}^{n} \int_{c_{j-1}}^{c_j} g'(x)dx \\
&= \sum_{j=1}^{n} \left[g(c_j^-) - g(c_{j-1}^+)\right] \\
&= \sum_{j=1}^{n} \left[g(c_j) - g(c_{j-1})\right] \\
&= (g(c_1) - g(c_0)) + (g(c_2) - g(c_1)) + \cdots + (g(c_n) - g(c_{n-1})) \\
&= g(c_n) - g(c_0) = g(b) - g(a) \; .
\end{aligned}$$

Proof of theorem. We use the claim to compute the Fourier coefficients of f'

$$a_0' = \frac{1}{\pi}\int_0^{2\pi} f'(x)dx = \frac{f(2\pi) - f(0)}{\pi} = 0$$

(since f is 2π-periodic). For a_n $(n \geq 1)$ we use integration by parts

$$\begin{aligned}
a_n' &= \frac{1}{\pi}\int_0^{2\pi} f'(x)\cos nx dx \\
&= \left(\frac{f(x)\cos nx}{\pi}\right)_{x=0}^{x=2\pi} + \frac{1}{\pi}\int_0^{2\pi} f(x) n \sin nx dx \\
&= \frac{n}{\pi}\int_0^{2\pi} f(x)\sin nx dx = n b_n
\end{aligned}$$

I leave it as an exercise for you to check that $b_n' = -n a_n$.

Example. Consider the triangular wave function of Example 1 of the previous section. It is defined on $[-\pi,\ \pi]$ by $f(x) = |x|$. This function is continuous on $\mathbb{R}$ and is 2π periodic. Furthermore its derivative $f'(x)$ is piecewise smooth: it is 1 for $0 < x < \pi$ and -1 for $-\pi < x < 0$. f' is the function of example 2 of the previous section. We have found

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\cos(2j+1)x}{(2j+1)^2}\,, \quad \forall x \in [-\pi,\ \pi] \; .$$

and by the Theorem we can differentiate term by term to get

$$f'(x) \;\sim\; \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{(2j+1)} \; .$$

4. Integration of Fourier Series

In general an antiderivative of a periodic function is not periodic. For example $f(x) = 1$ is periodic (of any period) but its antiderivatives $F(x) = x + C$ are not periodic. The following lemma gives a necessary and sufficient condition for an antiderivative to be periodic.

Lemma. *Let f be a T-periodic and piecewise continuous function on $\mathbb{R}$. The antiderivative F of f defined by*

$$F(x) = \int_0^x f(t)dt$$

is T-periodic if and only if

$$\int_0^T f(t)dt = 0\ .$$

Proof. Suppose that f satisfies $\int_0^T f(t)dt = 0$. We need to show that $F(x+T) = F(x)$. We have

$$F(x+T) - F(x) = \int_0^{x+T} f(t)dt - \int_0^x f(t)dt = \int_x^{x+T} f(t)dt = \int_0^T f(t)dt = 0.$$

Conversely, if the antiderivative is T-periodic, then

$$0 = F(T) - F(0) = \int_0^T f(t)dt\ .$$

We get the following result about term by term integration of Fourier series.

Theorem. *Let f be a piecewise smooth and 2π-period function satisfying*

$$\int_0^{2\pi} f(x)dx = 0 \quad \Leftrightarrow \quad a_0 = 0\ .$$

Consider the antiderivative of f defined by $F(x) = \int_0^x f(t)dt$. Then the Fourier series is F is obtained from that of f by termwise integration. That is, if

$$f(x)\ \sim\ \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\ ,$$

Then,

$$F(x) = A_0 + \sum_{n=1}^{\infty} \frac{-b_n}{n} \cos nx + \frac{a_n}{n} \sin nx\ ,$$

where $A_0 = \sum_{n=1}^{\infty} \frac{b_n}{n}$.

Proof. The antiderivative F is continuous and it is also 2π-periodic (see Lemma). Since $F' = f$ is piecewise smooth, then we can apply the previous Theorem about differentiation of Fourier series. Consider the Fourier series of F:

$$F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx\ .$$

Then

$$F'(x)\ \sim\ \sum_{n=1}^{\infty} nB_n \cos nx - nA_n \sin nx$$

and so

$$(nB_n = a_n,\quad -nA_n = b_n)\ \Leftrightarrow\ (A_n = \frac{-b_n}{n},\quad B_n = \frac{a_n}{n})$$

The coefficient A_0 can be found by using $F(0) = 0$ and equating it to the value of the Fourier series at $x = 0$.

Example. We start with the rectangular wave function of example 2 of the previous section: f is defined by $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$ and f is 2π-periodic function. We have found

$$f(x) \sim \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)} .$$

Since $a_0 = 0$, we can integrate term by term to obtain the Fourier series of $F(x) = \int_0^x f(t)dt$

$$F(x) = A_0 - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2} , \quad \forall x \in [-\pi,\ \pi] .$$

with $A_0 = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}$. In fact $F(x) = |x|$ for $|x| \leq \pi$ and $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|dx = \frac{\pi}{2}$. We have recovered again

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2} , \quad \forall x \in [-\pi,\ \pi] .$$

For the 2π-periodic function $F(x) = |x|$ on $[-\pi,\ \pi]$, termwise integration of its Fourier series is not a pure trigonometric series but will contain an extra term, a contribution from A_0, since $\int_{-\pi}^{\pi} F(x)dx \neq 0$. More precisely, for $0 < x < \pi$, we have

$$\int_0^x tdt = \int_0^x \frac{\pi}{2}dt - \frac{4}{\pi}\sum_{j=0}^{\infty}\int_0^x \frac{\cos(2j+1)t}{(2j+1)^2}dt ,$$
$$\frac{x^2}{2} = \frac{\pi x}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)^3}$$

Thus, for $0 < x < \pi$, we have

$$\frac{x^2}{2} - \frac{\pi x}{2} = -\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)^3} .$$

The series $-\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)^3}$ is the Fourier series of the 2π-periodic function $G(x)$ given on the interval $[-\pi,\ \pi]$ by

$$G(x) = \int_0^x F(t)dt - A_0 x = \int_0^x |t|dt - \frac{\pi x}{2} = \begin{cases} (x^2 - \pi x)/2 & \text{if } 0 < x < \pi \\ (-x^2 - \pi x)/2 & \text{if } -\pi < x < 0 \end{cases}$$

In fact this allows us to obtain an expansion of x^2 over $[0,\ \pi]$ by using the series for x and for $(x^2-\pi x)/2$. We have

$$\begin{aligned} x^2 &= 2\frac{x^2-\pi x}{2}+\pi x \\ &= 2\left(-\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)^3}\right)+\pi\left(\frac{\pi}{2}-\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2}\right) \\ &= \frac{\pi^2}{2}-4\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2}+\frac{2\sin(2j+1)x}{\pi(2j+1)^3}. \end{aligned}$$

5. Uniform convergence of Fourier series

A sequence of functions $g_n(x)$ defined on an interval I is said to converge *uniformly* to a function $g(x)$ on I if the following holds.

$$\forall\epsilon>0,\ \exists N\in\mathbb{Z}^+,\ \ |g_n(x)-g(x)|<\epsilon,\ \ \forall x\in I,\ \ \forall n\geq N.$$

This means that for any given $\epsilon>0$, we can find N that depends only on ϵ so that

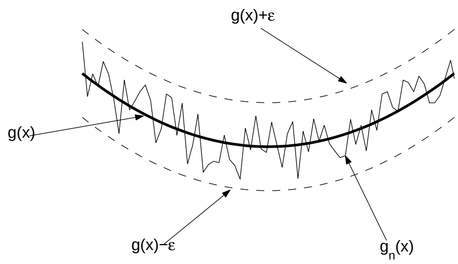

Figure 6. *Graph of $g_n(x)$ between those of $g(x)\pm\epsilon$*

$g_n(x)$ is within ϵ from $g(x)$ for all $x\in I$ and for all $n>N$.

A sequence of functions $h_n(x)$ defined on an interval I is said to converge (pointwise) to a function $h(x)$ on I if the following holds.

$$\forall x\in I,\ \ \forall\epsilon>0,\ \exists N\in\mathbb{Z}^+,\ \ |h_n(x)-h(x)|<\epsilon,\ \ \forall n\geq N.$$

In the pointwise convergence N depends on ϵ and on x.

The uniform convergence is stronger than pointwise convergence. A particular consequence of the uniform convergence is the following. If each g_n is continuous in an interval I and if $g_n \longrightarrow g$ uniformly in I, then g is also continuous.

A series of function $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function f on an interval I is the sequence of partial sums $s_n(x)=\sum_{j=1}^{n} f_j(x)$ converges uniformly to $f(x)$.

The following two propositions give sufficient conditions for the uniform convergence of Fourier series.

Proposition 1. *Let*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

be the Fourier series of a piecewise smooth and 2π-periodic function f. If $\sum_{n=1}^{\infty}(|a_n|+|b_n|) < \infty$, then the Fourier series converges uniformly.

Since the partial sums of the Fourier series are continuous, the proposition implies that the limit of the Fourier series is a continuous function. Thus the function f is continuous everywhere except possibly at removable discontinuities. This means $f(x^+) = f(x^-)$ everywhere. Note also that if f has a jump discontinuity at x_0 (i.e. $f(x_0^+) \neq f(x_0^-)$), then the Fourier series of f does not converge uniformly on any interval containing x_0.

Proposition 2. *If a function f is continuous in $\mathbb{R}$, is piecewise smooth, and 2π-periodic, then its Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

converges uniformly to f on $\mathbb{R}$.

6. Exercises

In the following exercises, a 2π-periodic function f is given on the interval $[-\pi,\ \pi]$. (a.) Find the Fourier series of f: (b.) Find the intervals where $f(x)$ is equal to its Fourier series: (c.) Determine whether the Fourier series converges uniformly

Exercise 1. $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 1 & \text{if } -\pi/2 < x < 0 \\ 0 & \text{if } \pi/2 < |x| < \pi \end{cases}$

Exercise 2. $f(x) = |\sin x|$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

Exercise 3. $f(x) = |\cos x|$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1}$.

Exercise 4. $f(x) = \cos^2 x$ (thing about a trig. identity)

Exercise 5. $f(x) = \sin^2 x$

Exercise 6. $f(x) = x^2$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

Exercise 7. $f(x) = x(\pi - |x|)$. Use the Fourier series to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}$.

Exercise 8. Use the Fourier series for x^2 that you found in exercise 6 to deduce the fourier series of $x^3 - \pi^2 x$ on $[-\pi,\ \pi]$ (use integration of Fourier series).

Exercise 9. Use the Fourier series you found in exercise 8. To deduce that

$$x^4 - 2\pi^2 x^2 = -\frac{7\pi^4}{15} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^4} \quad \text{for } -\pi < x < \pi\ .$$

Deduce the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Exercise 10. Suppose that $f(x)$ has Fourier series $\displaystyle\sum_{n=1}^{\infty} \mathrm{e}^{-n^2} \sin nx$. Find the Fourier series of $f'(x)$ and the Fourier series of $f''(x)$ (justify your answer).

Appendix

In this appendix, we prove Propositions 1 and 2 about uniform convergence of Fourier series. Given a series $\sum_n f_n(x)$ of functions, a practical test for uniform convergence is the following.

Weierstrass M-Test. *Given a series of functions $\sum_n f_n(x)$ on an interval I. If there is a sequence of real numbers $M_n \geq 0$ such that*

$$|f_n(x)| \leq M_n\,, \quad \forall x \in I,\ \forall n \in \mathbb{Z}^+$$

and if $\displaystyle\sum_{n=1}^{\infty} M_n < \infty$ then the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I.

Schwartz Inequalities. *Given series of real numbers $\displaystyle\sum_{n=1}^{\infty} \alpha_n$ and $\displaystyle\sum_{n=1}^{\infty} \beta_n$ so that $\displaystyle\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\displaystyle\sum_{n=1}^{\infty} \beta_n^2 < \infty$, then*

$$\left|\sum_{n=1}^{\infty} \alpha_n \beta_n\right| \leq \left(\sum_{n=1}^{\infty} \alpha_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \beta_n^2\right)^{1/2}.$$

Given piecewise continuous functions f and g on an interval $[a,\ b]$, then we have

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b f(x)^2 dx\right)^{1/2} \left(\int_a^b g(x)^2 dx\right)^{1/2}$$

Proof. We prove the first inequality and leave the second as an exercise. The proof is based on the following observation: if A, B, and C are real constants such that $Ax^2 + 2Bx + C \geq 0$, $\forall x \in \mathbb{R}$, then necessarily
$B^2 - AC \leq 0$.

Now, let $N \in \mathbb{Z}^+$ and define A_N, B_N, and C_N by

$$A_N = \sum_{n=1}^{N} \alpha_n^2, \quad B_N = \sum_{n=1}^{N} \alpha_n \beta_n, \quad \text{and} \quad C_N = \sum_{n=1}^{N} \beta_n^2\,.$$

For $x \in \mathbb{R}$, we have

$$\sum_{n=1}^{N} (x\alpha_n + \beta_n)^2 = x^2 \sum_{n=1}^{N} \alpha_n^2 + 2x \sum_{n=1}^{N} \alpha_n \beta_n + \sum_{n=1}^{N} \beta_n^2 \geq 0.$$

Thus,

$$A_N x^2 + 2B_N x + C_N \geq 0, \qquad \forall x \in \mathbb{R}$$

and the observation implies that $B_N^2 \le A_N C_N$. Since by hypothesis, $\lim_{N\to\infty} A_N$ and $\lim_{N\to\infty} B_N$ are finite, we get (after letting $N \to \infty$)

$$\left(\sum_{n=1}^{\infty} \alpha_n \beta_n\right)^2 \le \sum_{n=1}^{\infty} \alpha_n^2 \sum_{n=1}^{\infty} \beta_n^2 \,.$$

The Schwartz inequality is obtained by taking the square root of the above inequality.

Bessel's inequality *Let f be a 2π-periodic and piecewise continuous function with Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, + \, b_n \sin nx \,.$$

Then

$$\frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2} \, \le \, \frac{1}{2\pi} \int_0^{2\pi} f(x)^2 dx \,.$$

Proof. Let $S_N f$ be the N-th partial sum of the Fourier series of f. We have

$$||f(x) - S_N f(x)||^2 = < f - S_N f, f - S_N f > = ||f||^2 - 2 < f, S_N f > + ||S_N f||^2 \,.$$

Now

$$f(x) S_N f(x) = \frac{a_0}{2} f(x) + \sum_{n=1}^{N} a_n \cos(nx) f(x) + b_n \sin(nx) f(x) \,.$$

To find $< f, S_N f >$, we integrate both sides from 0 to 2π and use the fact that

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx, \qquad \left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{\pi} \int_0^{2\pi} f(x) \left\{ \begin{matrix} \cos nx \\ \sin nx \end{matrix} \right. \, dx$$

to obtain

$$< f, S_N f > = 2\pi \left(\frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \right) \,.$$

To find $||S_N f||^2$, we use the orthogonality of the trigonometric system to obtain

$$\begin{aligned} \int_0^{2\pi} S_N f(x)^2 dx \quad &= 2\pi \frac{a_0^2}{4} + \sum_{n=1}^{N} a_n^2 || \cos nx||^2 + b_n^2 || \sin nx||^2 \\ &= 2\pi \left(\frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \right) \,. \end{aligned}$$

These equalities imply that

$$0 \le ||f(x) - S_N f(x)||^2 = ||f||^2 - 2\pi \left(\frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \right)$$

and then

$$\frac{a_0^2}{4} + \sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{2} \, \le \, \frac{1}{2\pi} ||f||^2 = \frac{1}{2\pi} \int_0^{2\pi} f(x)^2 dx \,.$$

The Bessel's inequality follows by letting $N \to \infty$.

Proof of Proposition 1. Suppose that the Fourier coefficients of f satisfy $\sum_n |a_n| + |b_n| < \infty$. We use the Weierstrass M-test to show that the Fourier series $(a_0/2) + \sum_n (a_n \cos nx + b_n \sin nx)$ converges uniformly on $\mathbb{R}$. For this, we just need to take $M_n = |a_n| + |b_n|$ and observe that

$$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \ .$$

Proof of Proposition 2. Suppose that f is continuous piecewise smooth and 2π-periodic. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \quad \text{and}$$
$$\sum_{n=1}^{\infty} a'_n \cos nx + b'_n \sin nx$$

be the Fourier series of f and of f'. We need to show that the first series converge uniformly. For this it is enough to show that the series $\sum_n |a_n| + |b_n| < \infty$ (the Weierstrass M-test again would imply uniform convergence).

We apply Bessel's inequality to f':

$$\sum_{n=1}^{\infty} ({a'}_n^2 + {b'}_n^2) \ < \ \frac{1}{\pi} \int_0^{2\pi} f'(x)^2 dx$$

We know that $a'_n = nb_n$ and $b'_n = -na_n$. Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} (|a_n| + |b_n|) \quad &= \sum_{n=1}^{\infty} \left(\frac{|a'_n|}{n} + \frac{|b'_n|}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n} \left(|a'_n| + |b'_n| \right) \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=1}^{\infty} (|a'_n| + |b'_n|)^2 \right)^{1/2} \quad \text{(Schwartz inequality)} \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} 2 \left(\sum_{n=1}^{\infty} (|a'_n|^2 + |b'_n|^2) \right)^{1/2} \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \frac{2}{\pi} \int_0^{2\pi} f'(x)^2 dx
\end{aligned}$$

This proves the uniform convergence of the Fourier series of f.