FOURIER SERIES PART III: APPLICATIONS

We extend the construction of Fourier series to functions with arbitrary periods, then we associate to functions defined on an interval $[0,\ L]$ Fourier sine and Fourier cosine series and then apply these results to solve BVPs.

1. Fourier series with arbitrary periods

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a piecewise continuous function with period $2p$ ($p > 0$). We would like to represent f by a trigonometric series. We can repeat what we did in the previous note, when we had $p = \pi$, and reach the sought representation. There is however a simple way of obtaining the same series by introducing the function

$$g(s) = f\left(\frac{ps}{\pi}\right) \quad \left(\Leftrightarrow\ f(x) = g\left(\frac{\pi x}{p}\right)\right) .$$

Note that since $f \in C_p^0(\mathbb{R})$, then $g \in C_p^0(\mathbb{R})$ and that since f is $2p$-periodic, then

$$g(s + 2\pi) = f\left(\frac{p(s + 2\pi)}{\pi}\right) = f\left(\frac{ps}{\pi} + 2p\right) = f\left(\frac{ps}{\pi}\right) = g(s).$$

That is, g is 2π-periodic. We can therefore associate a Fourier series to g:

$$g(s) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos ns + b_n \sin ns .$$

In terms of the function f, we have the association

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} .$$

The coefficients are given by

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)ds = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ps}{\pi}\right) ds = \frac{1}{p}\int_{-p}^{p} f(x)dx \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\cos nsds = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ps}{\pi}\right)\cos(ns)ds = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right) dx \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\sin nsds = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ps}{\pi}\right)\sin(ns)ds = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right) dx
\end{aligned}$$

The fundamental convergence theorem (Fourier theorem) states

Theorem. *Let f be a $2p$-periodic and piecewise smooth function on $\mathbb{R}$. Then*

$$f_{av}(x) = \frac{f(x^+) + f(x^-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} ,$$

where the coefficients are given by

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx ,$$

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and for $n \in \mathbb{Z}^+$,

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right) dx\,, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right) dx\,.$$

Again if f is continuous at x_0, then $f(x_0)$ is equal to its Fourier series:

$$f(x_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x_0}{p} + b_n \sin\frac{n\pi x_0}{p}\;.$$

We also have uniform convergence with the additional condition that f is continuous on $\mathbb{R}$. More precisely,

Theorem. *Let f be a $2p$-periodic and piecewise smooth function on $\mathbb{R}$. Suppose that f is continuous in $\mathbb{R}$, then*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p} \qquad \forall x \in \mathbb{R}$$

and the convergence is uniform.

Analogous results hold for termwise differentiation of Fourier series and termwise integration.

Example. Consider the function $f(x)$ that is $2p$-periodic and is given on the interval $[-p,\ p]$ by

$$f(x) = \begin{cases} \dfrac{-2Hx}{p} + H & \text{if } 0 \le x < p/2\;; \\ \dfrac{2Hx}{p} + H & \text{if } -p/2 \le x < 0\;; \\ 0 & \text{if } p/2 \le |x| \le p \end{cases}$$

where H is a positive constant.

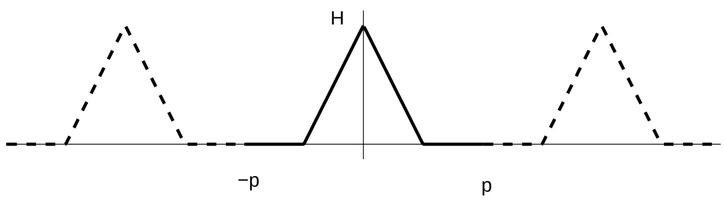


FIGURE 1. *Graph of function of example*

The function f is continuous on $\mathbb{R}$ and is piecewise smooth. It is also an even function (hence its b_n Fourier coefficients are all zero). The Fourier coefficients of f are

$$a_0 = \frac{2}{p}\int_0^p f(x)dx = \frac{2H}{p}\int_0^{p/2}\left(\frac{-2x}{p}+1\right)dx = \frac{H}{2}$$

and for $n = 1,\ 2,\ 3,\ \cdots$ we have

$$\begin{aligned} a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}dx = \frac{2H}{p}\int_0^{p/2}\left(\frac{-2x}{p}+1\right)\cos\frac{n\pi x}{p}dx \\ &= \frac{2H}{p}\left[\left(\frac{-2x}{p}+1\right)\frac{p}{n\pi}\sin\frac{n\pi x}{p}\right]_0^{p/2} + \frac{4H}{pn\pi}\int_0^{p/2}\sin\frac{n\pi x}{p}dx \\ &= \frac{4H}{\pi^2 n^2}\left(1-\cos\frac{n\pi}{2}\right) \end{aligned}$$

Since the function f is continuous and piecewise smooth, we have

$$f(x) = \frac{H}{4} + \frac{4H}{\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi/2)}{n^2}\cos\frac{n\pi x}{p} \qquad \forall x\in\mathbb{R}$$

Furthermore, the convergence is uniform.

We can apply termwise differentiation to obtain the Fourier series of the derivative f' (where $f'(x) = -2H/p$ for $0 < x < p/2$, $f'(x) = 2H/p$ for $-p/2 < x < 0$ and $f'(x) = 0$ for $p/2 < |x| < p$). We have

$$f'(x) \ \sim\ -\frac{4H}{\pi p}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi/2)}{n}\sin\frac{n\pi x}{p}$$

2. Parseval's Identity

Parseval's identity is a sort of a generalized *pythagorean theorem* in the space of functions.

Theorem. (Parseval's Identity) *Let f be $2p$-periodic and piecewise continuous with Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\ .$$

Then

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{1}{2p}\int_{-p}^{p} f(x)^2dx\ .$$

Proof. We will prove the identity when f is continuous and piecewise smooth. In this case the Fourier series (equals f),is uniformly convergent, and termwise integration is allowed. We have

$$\begin{aligned} f(x)^2 &= \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right) f(x) \\ &= \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty} a_n f(x)\cos\frac{n\pi x}{p} + b_n f(x)\sin\frac{n\pi x}{p}\ . \end{aligned}$$

We integrate from $-p$ to p and divide by $2p$. The term by term integration gives

$$\begin{aligned} \frac{1}{2p}\int_{-p}^{p} f(x)^2dx &= \frac{a_0}{2}\frac{1}{2p}\int_{-p}^{p} f(x)dx + \\ &+ \sum_{n=1}^{\infty} a_n\frac{1}{2p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx + b_n\frac{1}{2p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx\ . \\ &= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) \end{aligned}$$

The Parseval's identity is used to approximate the average error when replacing a given function f by its N-th Fourier partial sum $S_N f$. Recall that

$$S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p} \ .$$

The *mean square error*, when replacing f by $S_N f$, is defined as the number E_N given by

$$E_N^2 = \frac{1}{p}\int_{-p}^{p} (f(x) - S_N f(x))^2 \, dx \ .$$

If we use Parseval's identity to the function $f - S_N f$, we find that

$$E_N^2 = \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)$$

Example. Consider the 2π-periodic function defined over $[-\pi,\ \pi]$ by $f(x) = 1$ for $0 < x < \pi$ and $f(x) = -1$ for $-\pi < x < 0$. The fourier series of f is $\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)x}{(2j+1)}$. We would like to find N so that the approximation

$$f(x) \approx \frac{4}{\pi}\sum_{2j+1\le N}\frac{\sin(2j+1)x}{(2j+1)}$$

guarantees that the mean square error is no more than 0.01. That is $E_N < 10^{-2}$. From the above discussion, we have

$$E_N^2 = \frac{16}{\pi^2}\sum_{j>(N-1)/2}\frac{1}{(2j+1)^2} \ .$$

We can estimate the last series by using the integral test (see figure). We have

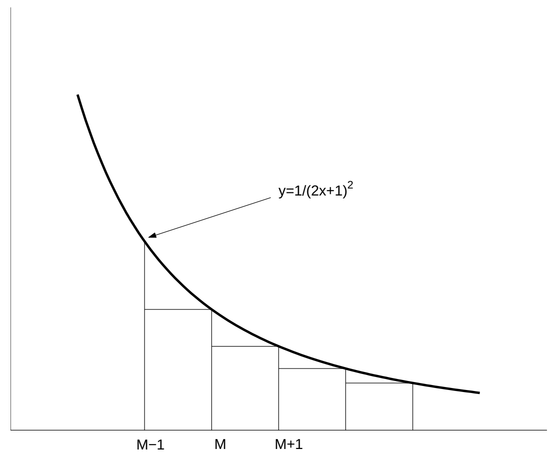


FIGURE 2. *Comparison of integral and sum*

$$\sum_{j=M}^{\infty}\frac{1}{(2j+1)^2} \le \int_{M-1}^{\infty}\frac{dx}{(2x+1)^2} = \frac{1}{2(2M-1)} \ .$$

Hence, it follows from the above calculations that

$$E_N^2 \leq \frac{16}{\pi^2 N}.$$

Therefore, in order to have $E_N < 0.01$, it is enough to take N so that $N > 1600/\pi^2$. That is $N = 163$.

Parseval's identity can also be used to evaluate series.

Example. We have seen that

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2} \qquad \forall x \in [-\pi,\ \pi]\ .$$

Parseval's identity gives

$$\frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|^2 dx = \frac{\pi^2}{3}\ .$$

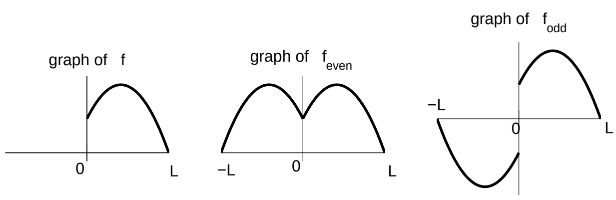
From this we get

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots = \frac{\pi^4}{96}\ .$$

3. Even and Odd Periodic Extensions

We would like to represent a function f given only on a an interval $[0,\ L]$ by a trigonometric series. For this we extend f to the interval $[-L,\ L]$ as either an even function or as an odd function then extend it to $\mathbb{R}$ as a periodic function with period $2L$. The Fourier series of this extension gives the sought representation of f. The even extension gives the *Fourier cosine* series of f and the odd extension gives the *Fourier sine* series of f.

More precisely, let f be a piecewise smooth function on the interval $[0,\ L]$. Let f_{even} and f_{odd} be, respectively, the even and the odd odd extensions of f to the interval $[-L,\ L]$. Now we extend f_{even} to $\mathbb{R}$ as a $2L$-periodic function F_{even} and

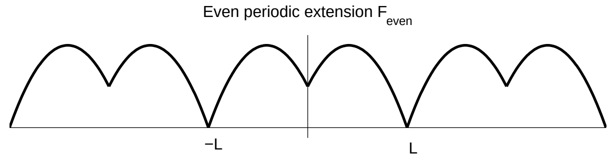


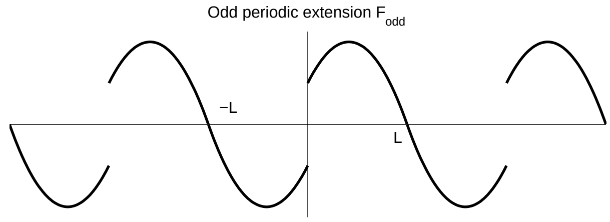
we extend f_{odd} to $\mathbb{R}$ as a $2L$-periodic function F_{odd}. Note

$$f(x) = F_{even}(x) = F_{odd}(x) \qquad \forall x \in [0,\ L]\ .$$

The Fourier series of F_{even} and F_{odd} are

$$F_{even}(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{and} \quad F_{odd}(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$





The Fourier coefficients are

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L F_{even}(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad n = 0,\ 1,\ 2,\ \cdots \\ b_n &= \frac{2}{L}\int_0^L F_{odd}(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad n = 1,\ 2,\ 3,\ \cdots \end{aligned}$$

This together with Fourier's Theorem give the following representations.

Theorem *Let f be a piecewise smooth function on the interval* $[0,\ L]$. *Then f has the following* Fourier cosine series *representation:* $\forall x \in (0,\ L)$

$$f_{av}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad \text{where } a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx\ .$$

In particular at each point x where f is continuous we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\ .$$

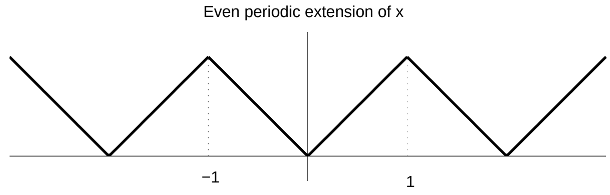
Theorem *Let f be a piecewise smooth function on the interval* $[0,\ L]$. *Then f has the following* Fourier sine series *representation:* $\forall x \in (0,\ L)$

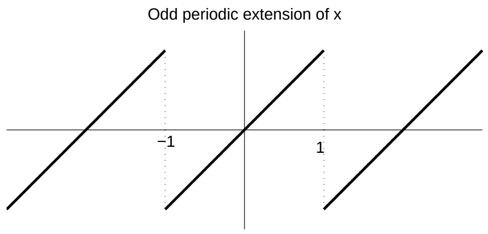
$$f_{av}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad \text{where } b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\ .$$

In particular at each point x where f is continuous we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\ .$$

Example 1. Let $f(x) = x$ on the interval $[0,\ 1]$. The 2-periodic even extension of f is the triangular wave function and the 2-periodic odd extension of f is the sawtooth function





If we use the even extension we get the Fourier cosine representation of x with coefficients

$$a_0 = \frac{2}{1}\int_0^1 x dx = 1$$

and for $n \geq 1$,

$$a_n = \frac{2}{1}\int_0^1 x \cos n\pi x dx = \frac{2((-1)^n - 1)}{n^2\pi^2}$$

Since $a_{2j} = 0$ and $a_{2j+1} = -\dfrac{4}{\pi^2(2j+1)^2}$, we get the Fourier cosine representation x over $[0,\ 1]$ as

$$x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{\cos[(2j+1)\pi x]}{(2j+1)^2} .$$

If we use the odd extension we get the Fourier sine representation of x with coefficients

$$b_n = \frac{2}{1}\int_0^1 x \sin n\pi x dx = \frac{2(-1)^{n+1}}{n\pi} .$$

The Fourier sine representation of x over the interval $[0, 1)$ is

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi x)}{n} .$$

Example 2. Find the Fourier cosine series of $f(x) = \sin x$ over the interval $[0,\ \pi]$.

We have

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi}\sin x dx = \frac{4}{\pi} ; \\ a_1 &= \frac{2}{\pi}\int_0^{\pi}\sin x\cos x dx = \frac{1}{\pi}\int_0^{\pi}\sin(2x) dx = 0 . \end{aligned}$$

To find a_n with $n \geq 2$, we use the identity

$$2\sin x\cos(nx) = \sin(n+1)x - \sin(n-1)x .$$

$$\begin{aligned} a_n &= \frac{2}{n}\int_0^{\pi} \sin x \cos(nx)dx = \frac{1}{\pi}\int_0^{\pi} (\sin(n+1)x - \sin(n-1)x)\,dx \\ &= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]_0^{\pi} \\ &= \frac{2((-1)^{n-1}-1)}{\pi(n^2-1)} \end{aligned}$$

We get the Fourier cosine of $\sin x$ on $[0,\ \pi]$ as

$$\sin x = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}-1}{n^2-1}\cos(nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos(2jx)}{(2j)^2-1}\ .$$

4. Heat Conduction in a Rod

Now we are in a position to solve BVPs with more general nonhomogeneous terms than the ones considered in Note 4. Consider the following BVP for the temperature function $u(x,t)$ in a rod of length L with initial temperature $f(x)$ and with ends kept at temperature 0.

$$\begin{array}{ll} u_t = ku_{xx} & 0 < x < L,\quad t > 0 \\ u(0,t) = 0,\ u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{array}$$

We can apply the method of separation of variables.

The homogeneous part of the BVP is

$$u_t = ku_{xx}, \quad u(0,t) = 0,\ u(L,t) = 0$$

We have seen that solutions $u(x,t) = X(x)T(t)$ (with separated variables) of the homogeneous part leads to the ODE problems

$$\left\{\begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{array}\right. \qquad T'(t) + k\lambda T(t) = 0\ .$$

The eigenvalues and eigenfunctions of the X-problem (Sturm-Liouville problem) are

$$\lambda_n = \nu_n^2, \quad X_n(x) = \sin(\nu_n x), \quad \text{where } \nu_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}^+$$

For each $n \in \mathbb{Z}^+$, the corresponding T-problem has a solution

$$T_n(t) = \mathrm{e}^{-k\nu_n^2 t}$$

and a solution of the homogeneous part with separated variables is $u_n(x,t) = T_n(t)X_n(x)$. The principle of superposition implies that any linear combination of these solutions is again a solution of the homogeneous part. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} C_n T_n(t) X_n(x) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-k\nu_n^2 t}\sin(\nu_n x)$$

solves *formally* the homogeneous part of the BVP. At the points (x,t) where the series converges and term by term differentiation (once in t and twice in x) is allowed, the function $u(x,t)$ defined by the series is a true solution of the homogeneous part. This will be addressed shortly.

For now let us find the constants C_n so that the formal solution solves also the nonhomogeneous condition $u(x,0) = f(x)$. That is, we would like the constants C_n so that

$$u(x,0) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-k\nu_n^2 0} \sin(\nu_n x) = f(x) \ .$$

Thus, after replacing ν_n by $n\pi/L$, we get

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \ .$$

This is the Fourier sine representation of the function f. Therefore the coefficients are given by

$$C_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx \,, \qquad n \in Z^+ \ .$$

Now we turn our attention to the series and verify that it indeed converges to a twice differentiable function u on $0 < x < L$ and $t > 0$ if f is piecewise smooth. For this we will use the Weierstrass M-test to prove uniform convergence. First, let $M > 0$ be an upper bound of f (i.e. $|f(x)| \leq M$ for every $x \in [0, L]$). We have

$$|C_n| \leq \frac{2}{L}\int_0^L |f(x)||\sin(\nu_n x)| dx \leq 2M \ .$$

It follows that for a given $t_0 > 0$, we have

$$\left| C_n \mathrm{e}^{-k\nu_n^2 t} \sin(\nu_n x) \right| \leq 2M \mathrm{e}^{-k\nu_n^2 t_0} \,, \qquad \forall t \geq t_0, \quad \forall x \in [0,\ L]$$

Since the numerical series $\sum_n 2M\mathrm{e}^{-k\nu_n^2 t_0}$ converges (use ratio or root tests), then it follows from the Weierstrass M-test that the series $\sum C_n \mathrm{e}^{-k\nu_n^2 t} \sin(\nu_n x)$ converges uniformly on the set $t \geq t_0$, $0 \leq x \leq L$. It follows at once that u is a continuous function. We can repeat the argument for the series giving u_t and the series giving u_{xx}. That is, the Weierstrass M-test shows that the series

$$u_t = \sum_{n=1}^{\infty} (-k\nu_n^2) C_n \mathrm{e}^{-k\nu_n^2 t} \sin(\nu_n x), \quad \text{and} \quad u_{xx} = \sum_{n=1}^{\infty} (-\nu_n^2) C_n \mathrm{e}^{-k\nu_n^2 t} \sin(\nu_n x)$$

converge uniformly on $t \geq t_0$, $x \in [0,\ L]$. We also have $u_t = k u_{xx}$. Consequently the function $u(x,t)$ given by the above series satisfies the complete BVP.

Example. Consider the BVP

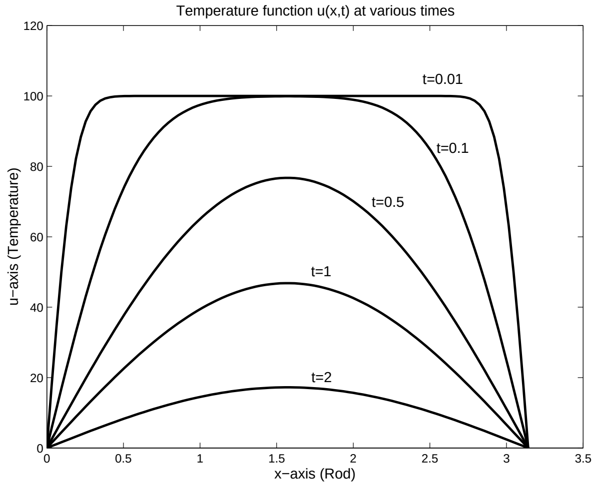
$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = 100 & 0 < x < \pi \end{array}$$

We have

$$C_n = \frac{2}{\pi}\int_0^{\pi} 100 \sin(nx) dx = \frac{200(1-(-1)^n)}{\pi n}$$

The solution of the BVP is therefore

$$u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \mathrm{e}^{-n^2 t} \sin(nx)$$



or equivalently

$$u(x,t) = \frac{400}{\pi} \sum_{j=0}^{\infty} \exp\left[-((2j+1)^2 t\right] \frac{\sin(2j+1)x}{2j+1} .$$

5. Wave Propagation in a String

Consider the BVP for the vibrations of a string with fixed ends.

$$\begin{array}{ll} u_{tt} = c^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \ u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < L \\ u_t(x,0) = g(x) & 0 < x < L \end{array}$$

Thus $u(x,t)$ represents the vertical displacement at time t of the point x on the string. The initial position and initial velocities of the string are given by the functions $f(x)$ and $g(x)$.

The homogeneous part (HP) of the BVP is

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = 0, \ u(L,t) = 0$$

The solutions $u(x,t) = X(x)T(t)$ (with separated variables) of the homogeneous part leads to the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} \qquad T''(t) + c^2 \lambda T(t) = 0 .$$

The eigenvalues and eigenfunctions of the X-problem (SL problem) are

$$\lambda_n = \nu_n^2, \quad X_n(x) = \sin(\nu_n x), \quad \text{where } \nu_n = \frac{n\pi}{L}, \quad n \in \mathbb{Z}^+$$

The corresponding T-problem has two independent solutions

$$T_n^1(t) = \cos(c\nu_n t) \quad \text{and} \quad T_n^2(t) = \sin(c\nu_n t) .$$

For each $n \in \mathbb{Z}^+$, we obtain solutions of (HP) with separated variables

$$\begin{array}{l} u_n^1(x,t) = T_n^1(t) X_n(x) = \cos(c\nu_n t) \sin(\nu_n x) \quad \text{and} \\ u_n^2(x,t) = T_n^2(t) X_n(x) = \sin(c\nu_n t) \sin(\nu_n x) . \end{array}$$

The principle of superposition implies that any linear combination of these solutions is again a solution of (HP). Thus,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n T_n^1(t) X_n(x) + B_n T_n^2 X_n(x) \\ &= \sum_{n=1}^{\infty} \left[A_n \cos(c\nu_n t) + B_n \sin(c\nu_n t)\right] \sin(\nu_n x) \end{aligned}$$

is a formal solution of (HP).

Now we use the nonhomogeneous conditions to find the coefficients A_n and B_n. First, we compute the (formal) derivative of u_t

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[c\nu_n B_n \cos(c\nu_n t) - c\nu_n A_n \sin(c\nu_n t)\right] \sin(\nu_n x) \ .$$

The conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ lead to

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} A_n \sin(\nu_n x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \quad \text{and} \\ g(x) &= \sum_{n=1}^{\infty} c\nu_n B_n \sin(\nu_n x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} B_n \sin \frac{n\pi x}{L} \ . \end{aligned}$$

These are the Fourier series sine representations of f and g on the interval $[0,\ L]$. Therefore

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad \text{and} \quad c\frac{n\pi}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

By using criteria for uniform convergence of Fourier series (Propositions 1 and 2 of Note 7), it can be shown that if f is continuous and piecewise smooth and if g is piecewise smooth, then the series defining u is uniformly convergent and $u(x,t)$ is a continuous function for $t \geq 0$ and $0 \leq x \leq L$. Moreover, we can show that if f, f', f'', g, and g' are continuous functions on $[0,\ L]$, the function $u(x,t)$ defined by the infinite series is twice differentiable in (x,t) and term by term differentiations in the series are valid. This give $u(x,t)$ as the (unique) solution of BVP.

Remark 1. Many concrete problems involve functions f that are only continuous and piecewise smooth. The series solution u is then only continuous. It is a 'continuous' solution of the BVP. The problem is understood in a more general sense: in the sense of *distributions* (a notion of generalized functions that is beyond the scope of this course).

Remark 2. In concrete application problems, to overcome the lack of differentiability of the series solution u, we can to within any degree of accuracy ϵ, replace the functions f and g by their truncated Fourier series $S_N f$ and $S_N g$ so that

$$||f - S_N f|| < \epsilon, \qquad ||g - S_N g|| < \epsilon \qquad \text{on } [0,\ L]$$

The functions $S_N f$ and $S_N g$ are infinitely differentiable and the corresponding solution u_N (the truncated series of u) is infinitely differentiable.

Remark 3. By using the principle of superposition, this BVP could have been split into two BVPs: BVP1 (plucked string)

$$\begin{array}{ll} v_{tt} = c^2 v_{xx} & 0 < x < L, \quad t > 0 \\ v(0,t) = 0,\ v(L,t) = 0 & t > 0 \\ v(x,0) = f(x) & 0 < x < L \\ v_t(x,0) = 0 & 0 < x < L \end{array}$$

and BVP2 (struck string)

$$\begin{array}{ll} w_{tt} = c^2 w_{xx} & 0 < x < L, \quad t > 0 \\ w(0,t) = 0,\ w(L,t) = 0 & t > 0 \\ w(x,0) = 0 & 0 < x < L \\ w_t(x,0) = g(x) & 0 < x < L \end{array}$$

The solutions to BVP1 and BVP2 are, respectively,

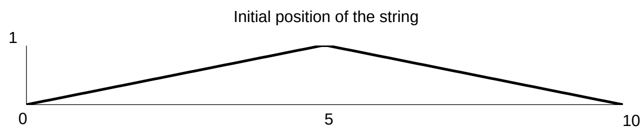
$$v = \sum_{n=1}^{\infty} A_n \cos(c\nu_n t) \sin(\nu_n x)$$
$$w = \sum_{n=1}^{\infty} B_n \sin(c\nu_n t) \sin(\nu_n x)$$

The solution to the original BVP is $u = v + w$.

Example 1. (Plucked string) Consider the BVP

$$\left\{ \begin{array}{ll} u_{tt} = 4v_{xx} & 0 < x < 10, \quad t > 0 \\ u(0,t) = 0,\ u(L,10) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < 10 \\ u_t(x,0) = 0 & 0 < x < 10 \end{array} \right.$$

where



$$f(x) = \left\{ \begin{array}{ll} x/5 & \text{if } 0 \le x \le 5 \\ (10-x)/5 & \text{if} 5 \le x \le 10 \end{array} \right.$$

For such an initial position we have ($B_n = 0$) and

$$\begin{aligned} A_n &= \frac{2}{10} \int_0^{10} f(x) \sin \frac{n\pi x}{10} dx \\ &= \frac{1}{25} \int_0^5 x \sin \frac{n\pi x}{10} dx + \frac{1}{25} \int_5^{10} (10-x) \sin \frac{n\pi x}{10} dx \\ &= \frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2} \end{aligned}$$

Hence, $A_{2j} = 0$ and $A_{2j+1} = \dfrac{8(-1)^j}{\pi^2(2j+1)^2}$. The series solution is

$$\begin{aligned} u(x,t) &= \frac{8}{\pi^2}\sum_{j=0}^{\infty}\cos((2j+1)t/5)\frac{(-1)^j\sin((2j+1)\pi x/10)}{(2j+1)^2} \\ &= \frac{8}{\pi^2}\left(\cos(t/5)\sin(x/10) - \frac{\cos(3t/5)\sin(3x/10)}{9} + \right. \\ &\quad \left. + \frac{\cos(5t/5)\sin(5x/10)}{25} - \frac{\cos(7t/5)\sin(7x/10)}{49} + \cdots\right) \end{aligned}$$

The individual components $u_n(x,t) = \cos(n\pi t/5)\sin(n\pi x/10)$ are called the *harmonics* or *modes of vibrations*. The function $u_n(x,t)$ is just a sine function in x being scaled by a cosine function in t with frequency $n/10$.

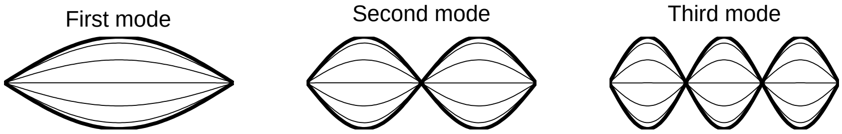


FIGURE 3. *The first three modes of vibrations of the plucked string at various times.*

Example 2. (Struck string) Consider the BVP

$$\begin{cases} u_{tt} = 4v_{xx} & 0 < x < 10, \quad t > 0 \\ u(0,t) = 0, \ u(L,10) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < 10 \\ u_t(x,0) = g(x) & 0 < x < 10 \end{cases}$$

where

$$g(x) = -1 \quad \text{if } 4 < x < 6, \quad \text{and } g(x) = 0 \quad \text{elsewhere} .$$

This time $A_n = 0$, and

$$\begin{aligned} \frac{n\pi}{5}B_n &= \frac{2}{10}\int_0^{10} g(x)\sin\frac{n\pi x}{10}dx \\ &= \frac{1}{\pi n}\left(\cos\frac{2n\pi}{5} - \cos\frac{3n\pi}{5}\right) \end{aligned}$$

Thus

$$B_n = \frac{5}{\pi^2 n^2}\left(\cos\frac{2n\pi}{5} - \cos\frac{3n\pi}{5}\right) = \frac{10}{\pi^2 n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{10}$$

The series solution is

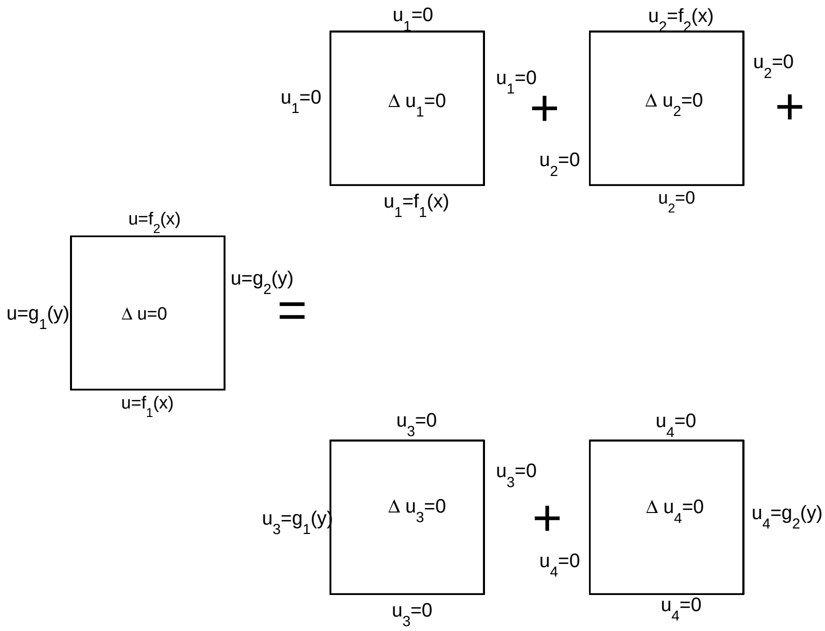
$$\begin{aligned} u(x,t) &= \frac{10}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{10}\sin\frac{n\pi t}{5}\sin\frac{n\pi x}{10} \\ &= \frac{10}{\pi^2}\left(\sin\frac{\pi}{10}\sin\frac{\pi t}{5}\sin\frac{\pi x}{10} - \frac{1}{9}\sin\frac{3\pi}{10}\sin\frac{3\pi t}{5}\sin\frac{3\pi x}{10} + \right. \\ &\quad \left. + \frac{1}{25}\sin\frac{5\pi}{10}\sin\frac{5\pi t}{5}\sin\frac{5\pi x}{10} - \frac{1}{49}\sin\frac{7\pi}{10}\sin\frac{7\pi t}{5}\sin\frac{7\pi x}{10} + \cdots\right) \end{aligned}$$

6. Problems Dealing with the Laplace Equation

Recall that the Dirichlet problem in a rectangle is to find a harmonic function u inside the rectangle whose values on the boundary are given. That is

$$\begin{array}{ll} \Delta u(x,y) = 0 & 0 < x < L, \quad 0 < y < H \\ u(x,0) = f_1(x), \quad u(x,H) = f_2(x) & 0 < x < L \\ u(0,y) = g_1(y), \quad u(L,y) = g_2(2) & 0 < y < H \end{array}$$

To solve this problem, we use the principle of superposition to decompose it into four simpler subproblems as in the figure.



We can find the solution $u(x,y)$ as

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y) \ .$$

Each of the subproblems can be solved by the method of separation of variables.

Now we indicate how to find $u_1(x,y)$. The solutions with separated variables $u_1(x,y) = X(x)Y(y)$ of the homogeneous part leads to the ODE problems

$$\left\{ \begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} Y''(y) - \lambda Y(y) = 0 \\ Y(H) = 0 \end{array} \right.$$

where λ is the separation constant. The X-problem is an SL-problem whose eigenvalues and eigenfunctions are

$$\lambda_n = \nu_n^2, \quad X_n(x) = \sin(\nu_n x), \quad \text{where} \quad \nu_n = \frac{n\pi}{L}, \ n \in \mathbb{Z}^+ \ .$$

For each λ_n, the corresponding ODE for the Y-problem has general solution $Y_n = A\cosh(\nu_n y) + B\sinh(\nu_n y)$. The boundary condition $Y(H) = 0$, implies that (up to a multiplicative constant), the solution of the Y-problem is

$$Y_n(y) = \sinh\left[\nu_n(H-y)\right] \ .$$

Hence, the solutions with separated variables of the homogeneous part of the u_1-problem are generated by

$$u_{1,n}(x,y) = \sinh\left[\nu_n(H-y)\right]\sin(\nu_n x) \qquad n \in \mathbb{Z}^+ \ .$$

A series solution of the u_1-problem is therefore

$$u_1(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left[\nu_n(H-y)\right]\sin(\nu_n x) \ .$$

Such a solution solves the nonhomogeneous condition $u(x,0) = f_1(x)$ if and only if

$$f_1(x) = \sum_{n=1}^{\infty} C_n \sinh(\nu_n H)\sin(\nu_n x) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi H}{L}\sin\frac{n\pi x}{L} \ .$$

Thus, $C_n \sinh(\nu_n H)$ is the n-th Fourier sine coefficient of f_1 over $[0,\ L]$:

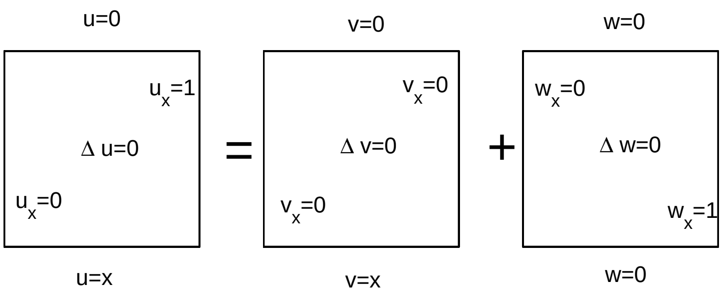
$$C_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx \ .$$

The functions u_2, u_3, and u_4 can be found in a similar way.

Example 1. Consider the following BVP with mixed boundary conditions.

$$\begin{array}{ll} \Delta u(x,y) = 0 & 0 < x < \pi, \quad 0 < y < 2\pi \\ u(x,0) = x, \quad u(x,2\pi) = 0 & 0 < x < \pi \\ u_x(0,y) = 0, \quad u_x(\pi,y) = 1 & 0 < y < 2\pi \end{array}$$

We decompose the problem as shown in the figure (see next page).



The method of separation of variables for the v-problem leads to the ODE problems

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \quad \text{and} \quad \begin{cases} Y''(y) - \lambda Y(y) = 0 \\ Y(2\pi) = 0 \end{cases}$$

where λ is the separation constant. The X-problem is an SL-problem whose eigenvalues and eigenfunctions are

$$\lambda_0 = 0, \qquad X_0(x) = 1$$

and for $n \in \mathbb{Z}^+$

$$\lambda_n = n^2, \quad X_n(x) = \cos(nx) \ .$$

For $\lambda_0 = 0$, the general solution of the ODE for the Y-problem is $Y(y) = A + By$ and in order to get $Y(2\pi) = 0$, we need $A = -2B\pi$. Thus $Y_0(y) = (2\pi - y)$ generates the solutions of the Y-problem For $\lambda_n = n^2$, the solutions of the Y-problem are generated by $Y_n(y) = \sinh\left[n(2\pi - y)\right]$.

The solutions with separated variables of the homogeneous part of the v-problem are therefore

$$v_0(x,y) = 2\pi - y \quad \text{and} \quad v_n(x,y) = \sinh[n(2\pi - y)]\cos(nx) \quad \text{for} \quad n \in \mathbb{Z}^+$$

The series solution is

$$v(x,y) = C_0(2\pi - y) + \sum_{n=1}^{\infty} C_n \sinh[n(2\pi - y)]\cos(nx) \ .$$

In order for such a series to solve the nonhomogeneous condition $v(x,0) = x$, we need to have

$$x = 2\pi C_0 + \sum_{n=1}^{\infty} C_n \sinh(2n\pi)\cos(nx) \ .$$

This is the Fourier cosine expansion of x over $[0,\ \pi]$. Hence,

$$2\pi C_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi \quad \Rightarrow \quad C_0 = \frac{1}{2}$$

and for $n \geq 1$

$$\sinh(2n\pi)C_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2\left((-1)^n - 1\right)}{\pi n^2} \ .$$

This gives

$$C_{2j} = 0 \quad \text{and} \quad C_{2j+1} = \frac{-4}{\pi(2j+1)^2 \sinh[2\pi(2j+1)]} \ .$$

The solution of the v-problem is

$$v(x,y) = \frac{2\pi - y}{4} - \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\sinh[(2j+1)(2\pi - y)]}{\sinh[2(2j+1)\pi]} \frac{\cos(2j+1)x}{(2j+1)^2} \ .$$

Now we solve the w-problem. The separation of variables for the homogeneous part leads to the ODE problems.

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(2\pi) = 0 \end{cases}$$

This time it is the Y-problem that is a Sturm-Liouville problem with eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2}{2^2}, \qquad Y_n(y) = \sin\frac{ny}{2}, \qquad n \in \mathbb{Z}^+ \ .$$

For each n, a generator of the solutions of the X-problem is

$$X_n(x) = \cosh\frac{nx}{2} \ .$$

The series solution of the w-problem is therefore

$$w(x,y) = \sum_{n=1}^{\infty} C_n \cosh\frac{nx}{2}\sin\frac{ny}{2} \ .$$

To find the coefficients C_n so that $w_x(\pi, y) \equiv 1$, we need

$$w_x(x,y) = \sum_{n=1}^{\infty} \frac{n}{2} C_n \sinh\frac{nx}{2}\sin\frac{ny}{2}$$

This gives

$$1 = \sum_{n=1}^{\infty} \frac{n}{2} C_n \sinh \frac{n\pi}{2} \sin \frac{ny}{2}$$

(the Fourier sine expansion of 1 over the interval $[0,\ 2\pi]$):

$$\frac{n}{2} C_n \sinh \frac{n\pi}{2} = \frac{2}{2\pi} \int_0^{2\pi} \sin \frac{ny}{2} dy = \frac{2\,(1-(-1)^n)}{n\pi} .$$

Equivalently,

$$C_{2j} = 0, \qquad C_{2j+1} = \frac{8}{\pi(2j+1)^2 \sinh\left[(2j+1)\pi/2\right]} .$$

The solution of the w-problem is therefore

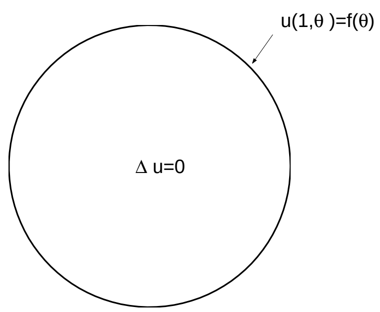
$$w(x,y) = \frac{8}{\pi} \sum_{j=0}^{\infty} \frac{\cosh\left[(2j+1)x/2\right]}{\sinh\left[(2j+1)\pi/2\right]} \frac{\sin\left[(2j+1)y/2\right]}{(2j+1)^2}$$

The solution u of the original problem is $u(x,y) = v(x,y) + w(x,y)$.

Example 2. Consider the Dirichlet problem in a disk (written in polar coordinates)

$$\begin{array}{ll} \Delta u(r,\theta) = 0 & r < 1, \quad \theta \in [0,\ 2\pi] \\ u(1,\theta) = f(\theta) & \theta \in [0,\ 2\pi] \end{array}$$

We take the function f to be given by



$$f(\theta) = \begin{cases} \sin\theta & \text{if } 0 \le \theta \le \pi \\ 0 & \text{if } \pi \le \theta \le 2\pi \end{cases}$$

Recall that the Laplace operator in polar coordinates is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

A solution with separated variables $u(r,\theta) = R(r)\Theta(\theta)$ leads to the ODEs

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad \text{and} \quad r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

where λ is the separation constant. Note that since $u(r,\theta+2\pi) = u(r,\theta)$, then the function Θ and also Θ' need to be 2π-periodic. Thus to the ODE for Θ we need to add $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. Hence, the Θ-problem is a periodic SL-problem whose eigenvalues and eigenfunctions are

$$\lambda_0 = 0, \qquad \Theta_0(\theta) = 1 ,$$

and for $n \in \mathbb{Z}^+$,

$$\lambda_n = n^2, \qquad \Theta_n^1(\theta) = \cos(n\theta), \quad \Theta_n^2(\theta) = \sin(n\theta) \ .$$

The ODE for the R-function is a Cauchy-Euler equation with characteristic equation $m^2 - \lambda = 0$. For $\lambda = \lambda_0 = 0$, the general solution of the R-equation are generated by

$$R_0^1(r) = 1 \quad \text{and} \quad R_0^2(r) = \ln r \ .$$

For $\lambda = \lambda_n$ (we have $m = \pm n$), the solutions of the R-equation are generated by

$$R_n^1(r) = r^n \quad \text{and} \quad R_n^2(r) = r^{-n} \ .$$

The solutions with separated variables of the Laplace equation $\Delta u = 0$ in the disk are therefore

$$1,\ \ln r,\ r^n \cos(n\theta),\ r^n \sin(n\theta),\ r^{-n}\cos(n\theta),\ r^{-n}\sin(n\theta) \ .$$

Since we looking for solutions that are bounded in the disk and since $\ln r$ and $r^{-n}\cos(n\theta)$ and $r^{-n}\sin(n\theta)$ are not bounded, then we will discard then when forming u

The series solution has the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right)$$

The initial condition $u(1,\theta) = f(\theta)$ leads to

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta) \ .$$

This is the Fourier series of f. I leave it as an exercise for you to verify that the Fourier series of f is

$$f(\theta) = \frac{1}{\pi} + \frac{1}{2}\sin\theta - \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{\cos(2j\theta)}{4j^2 - 1} \ .$$

The solution of the Dirichlet problem is

$$u(r,\theta) = \frac{1}{\pi} + \frac{r}{2}\sin\theta - \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{r^{2j}\cos(2j\theta)}{4j^2 - 1}$$

7. Exercises

Exercise 1. (a) Find the Fourier series of the function with period 4 that is defined over $[-2,\ 2]$ by $f(x) = \dfrac{4 - x^2}{2}$.

(b) Use Parseval's equality to evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

(c) Use the integral test to estimate the mean square error E_N when replacing f by its truncated Fourier series $S_N f$.

(d) Find N so that $E_N \le 0.01$ and then find N so that $E_N \le 0.001$

Exercise 2. (a) Find the Fourier series of the function with period 4 that is defined over $[-2,\ 2]$ by

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \le x \le 2 \\ 1 + x & \text{if } -2 \le x \le 0 \end{cases}$$

(b) Use Parseval's equality to evaluate the series $\sum_{j=0}^{\infty} \frac{1}{(2j+1)^4}$.

(c) Use the integral test to estimate the mean square error E_N when replacing f by its truncated Fourier series $S_N f$.

(d) Find N so that $E_N \leq 0.01$ and then find N so that $E_N \leq 0.001$

Exercise 3. Find the Fourier sine series of $f(x) = \cos x$ over $[0,\ \pi]$ (What is the Fourier cosine series of $\cos x$ on $[0,\ \pi]$?)

Exercise 4. Find the Fourier cosine series of $f(x) = \sin x$ over $[0,\ \pi]$ (What is the Fourier sine series of $\sin x$ on $[0,\ \pi]$?)

Exercise 5. Find the Fourier cosine series of $f(x) = x^2$ over $[0,\ 1]$.

Exercise 6. Find the Fourier sine series of $f(x) = x^2$ over $[0,\ 1]$.

Exercise 7. Find the Fourier cosine series of $f(x) = x \sin x$ over $[0,\ \pi]$.

Exercise 8. Find the Fourier sine series of $f(x) = x \sin x$ over $[0,\ \pi]$.

Exercise 9. Solve the BVP

$$\begin{cases} u_t = u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < 2 \end{cases}$$

where

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

Exercise 10. Solve the BVP

$$\begin{cases} u_t = u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = u(2,t) = 0, & t > 0 \\ u(x,0) = \cos(\pi x), & 0 < x < 2 \end{cases}$$

Exercise 11. Solve the BVP

$$\begin{cases} u_t + u = (0.1)u_{xx}, & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t > 0 \\ u(x,0) = \sin x, & 0 < x < 2 \end{cases}$$

Exercise 12. Consider the BVP modeling heat propagation in a rod where the end points are kept at constant temperatures T_1 and T_2:

$$\begin{cases} u_t = ku_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = T_1,\ u(L,t) = T_2, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

Since T_1 and T_2 are not necessarily zero, we cannot apply directly the method of eigenfunctions expansion. To solve such a problem, we can proceed as follows.

1. Find a function $\alpha(x)$ (independent on time t) so that

$$\alpha''(x) = 0, \quad \alpha(0) = T_1\ \alpha(L) = T_2\ .$$

2. Let $v(x,t) = u(x,t) - \alpha(x)$. Verify that if $u(x,t)$ solves the given BVP, then $v(x,t)$ solves the following problem

$$\begin{cases} v_t = kv_{xx}, & 0 < x < L, \quad t > 0 \\ v(0,t) = 0, \ v(L,t) = 0, & t > 0 \\ v(x,0) = f(x) - \alpha(x), & 0 < x < L \end{cases}$$

The v-problem can be solved by the method of separation of variables. The solution u of the original problem is therefore $u(x,t) = v(x,t) + \alpha(x)$.

Exercise 13. Apply the method of described in Exercise 12 to solve the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = T_1, \ u(2,t) = T_2, & t > 0 \\ u(x,0) = f(x), & 0 < x < 2 \end{cases}$$

in the following cases
1. $T_1 = 100$, $T_2 = 0$, $f(x) = 0$.
2. $T_1 = 100$, $T_2 = 100$, $f(x) = 0$.
3. $T_1 = 0$, $T_2 = 100$, $f(x) = 50x$.

In problems 14 to 16, solve the wave propagation problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \ u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), \ u_t(x,0) = g(x) & 0 < x < L \end{cases}$$

Exercise 14. $c = 1$, $L = 2$, $f(x) = 0$, $g(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}$

Exercise 15. $c = 1/\pi$, $L = 2$, $f(x) = \sin x$, $g(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \end{cases}$

Exercise 16. $c = 2$, $L = \pi$, $f(x) = x \sin x$, $g(x) = \sin(2x)$.

In exercises 17 to 19, solve the wave propagation problem with damping

$$\begin{cases} u_{tt} + 2au_t = c^2 u_{xx}, & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \ u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), \ u_t(x,0) = g(x) & 0 < x < L \end{cases}$$

Exercise 17. $c = 1$, $a = .5$, $L = \pi$, $f(x) = 0$, $g(x) = x$

Exercise 18. $c = 4$, $a = \pi$, $L = 1$, $f(x) = x(1-x)$, $g(x) = 0$.

Exercise 19. $c = 1$, $a = \pi/6$, $L = 2$, $f(x) = x\sin(\pi x)$, $g(x) = 1$.

In exercises 20 to 22, solve the Laplace equation $\Delta u(x,y) = 0$ inside the rectangle $0 < x < L$, $0 < y < H$ subject the the given boundary conditions.

Exercise 20. $L = H = \pi$, $u(x,0) = x(\pi - x)$, $u(x,\pi) = 0$, $u(0,y) = u(\pi,y) = 0$.

Exercise 21. $L = \pi$, $H = 2\pi$, $u(x,0) = 0$, $u(x,2\pi) = x$, $u_x(0,y) = \sin y$, $u_x(\pi,y) = 0$.

Exercise 22. $L = H = 1$, $u(x,0) = u(x,1) = 0$, $u(0,y) = 1$, $u(1,y) = \sin y$.

Exercise 23. Solve the Laplace equation $\Delta u(r,\theta) = 0$ inside the semicircle of radius 2 ($0 < r < 2$, $0 < \theta < \pi$) subject to the boundary conditions

$$u(r,0) = u(r,\pi) = 0 \quad (0 < r < 2) \quad \text{and} \quad u(2,\theta) = \theta(\pi - \theta) \quad (0 < \theta < \pi)$$

Exercise 24. Solve the Laplace equation $\Delta u(r,\theta) = 0$ inside the semicircle of radius 2 ($0 < r < 2$, $0 < \theta < \pi$) subject to the boundary conditions

$$u_\theta(r,0) = u_\theta(r,\pi) = 0 \quad (0 < r < 2) \quad \text{and} \quad u(2,\theta) = \theta(\pi - \theta) \quad (0 < \theta < \pi)$$

Exercise 25. Solve the Laplace equation $\Delta u(r,\theta) = 0$ inside the quarter of a circle of radius 2 ($0 < r < 2$, $0 < \theta < \pi/2$) subject to the boundary conditions

$$u(r,0) = u(r,\pi/2) = 0 \quad (0 < r < 2) \quad \text{and} \quad u(2,\theta) = \theta \quad (0 < \theta < \pi/2)$$