STURM-LIOUVILLE PROBLEMS: GENERALIZED FOURIER SERIES

1. Regular Sturm-Liouville Problem

The method of separation of variables to solve boundary value problems leads to ordinary differential equations on intervals with conditions at the endpoints of the intervals. For example heat propagation in a rod of length L whose end points are kept at temperature 0 leads to the ODE problem

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \quad X(L) = 0.$$

If the endpoints are insulated, the ODE problem becomes

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = 0, \quad X'(L) = 0.$$

These are examples of Sturm-Liouville problems.

A *regular Sturm-Liouville problem* (SL problem for short) on an interval $[a,\ b]$ is a second order ODE problem, with endpoints conditions, of the form

$$\begin{aligned} &(p(x)y')' + [\lambda r(x) - q(x)]\, y = 0 \\ &\alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ &\beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned} \tag{1}$$

where

- p, q and r are continuous functions on $[a,\ b]$ such that
$$p(x) > 0 \ \text{ and } \ r(x) > 0 \qquad \forall x \in [a,\ b]$$
- α_1, α_2, β_1, and β_2 are constants such that α_1, α_2 are not both zero, and β_1, β_2 are not both zero.
- $\lambda \in \mathbb{R}$ is a parameter

Remark 1. The second order ODE of the SL-problem (1) is in *self-adjoint* form (suitable for certain manipulations). Most second order linear ODEs can be transformed into a self-adjoint form. Consider the second order ODE

$$A(x)y'' + B(x)y' + C(x)y = 0 \quad \Leftrightarrow \quad y'' + \frac{B}{A}y' + \frac{C}{A}y = 0$$

with $A(x) > 0$ on an interval I. Let $p(x)$ be defined by

$$p(x) = \exp\left(\int \frac{B(x)}{A(x)}dx\right).$$

Then $p' = (B/A)p$ and $py'' + p(B/A)y' = (py')'$. Hence, if we multiply the last ODE by p we obtain

$$py'' + p(B/A)y' + p(C/A)y = 0 \quad \Leftrightarrow \quad (py')' + p(C/A)y = 0$$

This last ODE is in self-adjoint form.

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Note that $y \equiv 0$ is a solution of the SL-Problem (1). It is the *trivial* solution. For most values of the parameter λ, problem (1) has only the trivial solution. An *eigenvalue* of the the SL-problem (1) is a value of λ for which a nontrivial solution exist. The nontrivial solution is called an *eigenfunction.* Note that if $y(x)$ solves (1), then so does any multiple $cy(x)$, where c is a constant. Thus if y_1 is an eigenfunction of (1) with eigenvalue λ_1, then any function $cy_1(x)$ is also an eigenfunction with eigenvalue λ_1. In fact the set of all eigenfunctions, corresponding to an eigenvalue λ, together with the zero function forms a vector space: the *eigenspace* of the eigenvalue.

Example 1. For the SL-problem

$$y''(x) + \lambda y(x) = 0, \qquad y(0) = y(L) = 0$$

we have $p(x) = 1$, $r(x) = 1$, $q(x) = 0$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 0$. This problem, that we have encountered several times already, has infinitely many eigenvalues $\lambda_n = (n\pi/L)^2$, and for each $n \in \mathbb{Z}^+$, the eigenspace is generated by the function $y_n(x) = \sin(n\pi x/L)$.

Example 2. The eigenvalues and eigenfunction of the SL-problem

$$y''(x) + \lambda y(x) = 0, \qquad y'(0) = y'(L) = 0$$

(here we have $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$) are: $\lambda_0 = 0$ with $y_0(x) = 1$, and for $n \in \mathbb{Z}^+$, $\lambda_n = (n\pi/L)^2$ with $y_n(x) = \cos(n\pi x/L)$.

Example 3. Consider the SL-problem

$$y''(x) + \lambda y(x) = 0, \qquad y(0) = 0, \quad y'(L) = 0 \ ,$$

(this time $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 0$, and $\beta_2 = 1$). To find the eigenvalues, we consider three cases depending on the values of λ.

If $\lambda < 0$, set $\lambda = -\nu^2$ with $\nu > 0$. In this case the general solution of the ODE is $y(x) = C_1 \sinh(\nu x) + C_2 \cosh(\nu x)$. In order for $y(0) = 0$, we need to have $C_1 = 0$, thus $y = C_1 \sinh(\nu x)$ and $y' = \nu C_1 \cosh(\nu x)$. To get $y'(L) = 0$, we need $\nu C_1 \cosh(\nu L) = 0$. Since $\cosh x > 0$ ($\forall x \in \mathbb{R}$)and since $\nu > 0$, then necessarily $C_1 = 0$ and then $y \equiv 0$. Thus $\lambda < 0$ cannot be an eigenvalue.

If $\lambda = 0$, then $y(x) = A + Bx$ is the general solution of the ODE and the boundary conditions give $A = B = 0$, and consequently $\lambda = 0$ is not an eigenvalue.

If $\lambda > 0$, we set $\lambda = \nu^2$ with $\nu > 0$. The general solution of the ODE is $y(x) = C_1 \sin(\nu x) + C_2 \cos(\nu x)$. The condition $y(0) = 0$ implies $C_2 = 0$. Then $y = C_1 \sin(\nu x)$ gives $y' = \nu C_1 \cos(\nu x)$ and the condition $y'(L) = 0$ gives $\nu C_1 \cos(\nu L) = 0$. In order to obtain a nontrivial solution, we need $C_1 \neq 0$. Consequently, $\cos(\nu L) = 0$. Hence

$$\nu L = \frac{\pi}{2} + j\pi = \frac{(2j+1)\pi}{2}, \qquad j = 0,\ 1,\ 2,\ \cdots$$

The eigenvalues and eigenfunctions of the SL-problem are therefore

$$\lambda_j = \left(\frac{(2j+1)\pi}{2}\right)^2, \quad y_j(x) = \sin\frac{(2j+1)\pi x}{2}, \qquad j = 0,\ 1,\ 2,\ \cdots$$

This is the typical situation for the general SL-problem (1). The set of eigenvalues forms an increasing sequence and for each eigenvalue, the corresponding eigenspace is generated by one function. More precisely, we have the following Theorem,

Theorem 1. *The set of eigenvalues of the SL-problem (1) forms an increasing sequence*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \text{with} \quad \lim_{j\to\infty} \lambda_j = \infty$$

and for each $j \in \mathbb{Z}^+$, the eigenspace of λ_j is one-dimensional: it is generated by one eigenfunction $y_j(x)$

The proofs of the Theorems of this Note will not given here but can be found in either *Ordinary Differential Equations* by G. Birkhoff and G. Rotta; *Methods of Mathematical Physics* by R. Courant and D. Hilbert; *A First Course in PDE* by H.F. Weinberger.

Example 4. Let us find the eigenvalues and eigenfunctions of the SL-problem

$$\begin{aligned} &(x^2y')' + \lambda y = 0 \quad 1 < x < 2 \\ &y(1) = y(2) = 0 \end{aligned}$$

Note that ODE can be written as $x^2y'' + 2xy' + \lambda y = 0$ which a Cauchy-Euler equation. Its characteristic equation is $m^2 + m + \lambda = 0$. The characteristic roots are $m_1 = \dfrac{-1-\sqrt{1-4\lambda}}{2}$ and $m_2 = \dfrac{-1+\sqrt{1-4\lambda}}{2}$. We distinguish three cases depending on the sign of $1 - 4\lambda$.

If $\lambda < 1/4$, then $1 - 4\lambda > 0$. Both characteristic roots m_1 and m_2 are distinct and real. The general solution of the ODE is

$$y = C_1x^{m_1} + C_2x^{m_2}$$

The endpoints conditions are

$$C_1 + C_2 = 0, \quad C_1 2^{m_1} + C_2 2^{m_2} = 0\ .$$

Since $m_1 \neq m_2$, the only solution is $C_1 = C_2 = 0$. Thus $\lambda < 1/4$ cannot be an eigenvalue.

If $\lambda = 1/4$, then $m_1 = m_2 = -1/2$. The general solution of the ODE is

$$y = Ax^{-1/2} + Bx^{-1/2}\ln x\ .$$

The endpoints conditions lead to $A = 0$ and then $B2^{-1/2}\ln 2 = 0$. We have again $y \equiv 0$ and $\lambda = 1/4$ is not an eigenvalue.

If $\lambda > 1/4$, set $1 - 4\lambda = -4\nu^2$ with $\nu > 0$. In this case the characteristic roots are $m_{1,2} = -\dfrac{-1}{2} \pm i\nu$. Two independent solutions of the ODE are x^{m_1} and x^{m_2}. These are however (conjugate) complex-valued functions. We need to take their real and imaginary part. For this recall that

$$x^{a+ib} = x^a x^{ib} = x^a \mathrm{e}^{ib\ln x} = x^a\cos(b\ln x) + x^a\sin(b\ln x)\ .$$

The general solution of the ODE is therefore

$$y(x) = x^{-1/2}(C_1\cos(\nu\ln x) + C_2\sin(\nu\ln x)) = \frac{C_1\cos(\nu\ln x) + C_2\sin(\nu\ln x)}{\sqrt{x}}\ .$$

The endpoints conditions give

$$y(1) = C_1 = 0, \quad y(2) = \frac{C_1\cos(\nu\ln 2) + C_2\sin(\nu\ln 2)}{\sqrt{2}} = 0\ .$$

Hence, $C_2 \sin(\nu \ln 2) = 0$. To obtain a nontrivial solution y, we need to have $\sin(\nu \ln 2) = 0$. Thus, $\nu = \dfrac{n\pi}{\ln 2}$ with $n \in \mathbb{Z}^+$. From $1 - 4\lambda = -4\nu^2$, we get the eigenvalues

$$\lambda_n = \frac{1}{4} + \nu_n^2 = \frac{1}{4} + \left(\frac{n\pi}{\ln 2}\right)^2, \qquad n \in \mathbb{Z}^+$$

and the corresponding eigenfunctions

$$y_n(x) = \frac{1}{\sqrt{x}} \sin \frac{n\pi \ln x}{\ln 2} .$$

2. Orthogonality of Eigenfunctions

In view of applying eigenfunctions of SL-problems

$$\begin{aligned} &(p(x)y')' + [\lambda r(x) - q(x)]\, y = 0 \\ &\alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ &\beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned}$$

to expand functions and solve BVP, we introduce the following inner product in the space $C_p^0[a,\ b]$ of piecewise continuous functions on the interval $[a,\ b]$. We define the *inner product with weight* $r(x)$ as follows: for $f, g \in C_p^0[a,\ b]$,

$$< f, g >_r = \int_a^b f(x)g(x)r(x)dx .$$

The *norm* $||f||_r$ of a function f is defined as

$$||f||_r = \sqrt{< f, f >_r} = \left(\int_a^b f(x)^2 r(x)\right)^{1/2} .$$

Two function $f, g \in C_p^0[a,\ b]$ are said to be *orthogonal* if $< f, g >_r = 0$. We are going to show that for a given SL-problem, eigenfunctions corresponding to distinct eigenvalues are orthogonal. More precisely, we have the following Theorem.

Theorem 2. *Let $y(x)$ and $z(x)$ be two eigenfunctions of the SL-problem corresponding to two distinct eigenvalues λ and μ. Then y and z are orthogonal with respect to the inner product $< \ ,\ >_r$. That is,*

$$< y, z >_r = \int_a^b y(x)z(x)r(x)dx = 0 .$$

Proof. By using $\mu r z = -(pz')' + qz$, we get

$$\begin{aligned} \mu \int_a^b y(x)z(x)r(x)dx &= \int_a^b y(x)\left[-(p(x)z'(x))' + q(x)z(x)\right] dx \\ &= -\int_a^b y(x)(p(x)z'(x))'dx + \int_a^b q(x)y(x)z(x)dx \end{aligned}$$

We use integration by parts for the first integral in the right hand side

$$\begin{aligned} \int_a^b y(x)(p(x)z'(x))'dx &= [y(x)p(x)z'(x)]_a^b - \int_a^b p(x)y'(x)z'(x)dx \\ &= [p(x)(y(x)z'(x) - y'(x)z(x))]_a^b + \\ &\quad + \int_a^b (p(x)y'(x))'z(x)dx \end{aligned}$$

Now we use the endpoints conditions to get

$$\begin{aligned}\beta_1(y(b)z'(b)-y'(b)z(b)) &= (\beta_1 y(b))z'(b)) - y'(b)(\beta_1 z(b)) \\ &= -\beta_2 y'(b)z'(b) + \beta_2 y'(b)z'(b) = 0\end{aligned}$$

Similarly,

$$\beta_2(y(b)z'(b)-y'(b)z(b)) = 0\ .$$

Since β_1 and β_2 are not both zero, then $y(b)z'(b) - y'(b)z(b) = 0$. An analogous argument gives $y(a)z'(a) - y'(a)z(a) = 0$. This imply that

$$\left[p(x)(y(x)z'(x) - y'(x)z(x))\right]_a^b = 0\ .$$

Therefore,

$$\int_a^b y(x)(p(x)z'(x))'dx = \int_a^b (p(x)y'(x))'z(x)dx$$

and then

$$\begin{aligned}\mu\int_a^b y(x)z(x)r(x)dx &= -\int_a^b (p(x)y'(x))'z(x)dx + \int_a^b q(x)y(x)z(x)dx \\ &= \int_a^b \left[-(p(x)y'(x))' + q(x)y(x)\right] z(x)dx \\ &= \lambda\int_a^b y(x)z(x)r(x)dx\ .\end{aligned}$$

Since $\lambda \neq \mu$, then necessarily

$$\int_a^b y(x)z(x)r(x)dx = 0\ .$$

3. Generalized Fourier Series

Consider the regular SL-problem

$$\begin{aligned}&(p(x)y')' + [\lambda r(x) - q(x)]\, y = 0 \\ &\alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ &\beta_1 y(b) + \beta_2 y'(b) = 0\end{aligned} \tag{2}$$

We know that the eigenvalues λ_j form an increasing sequence and $\lim_{j\to\infty}\lambda_j = \infty$ and for each j, the eigenspace has dimension 1 (Theorem 1). Let $y_j(x)$ be an eigenfunction corresponding to the eigenvalue λ_j. By analogy with Fourier series, we are going to associates to each piecewise continuous function f on $[a,\ b]$ a series in the eigenfunctions y_j:

$$f(x) \sim \sum_{j=1}^{\infty} c_j y_j(x)$$

where the coefficients c_j are given by

$$c_j = \frac{< f, y_j >_r}{||y_j||_r^2} = \frac{\int_a^b f(x)y_j(x)r(x)dx}{\int_a^b y_j^2(x)r(x)dx} \tag{3}$$

The following Theorem says that the system of eigenfunctions $\{y_j\}_j$ is *complete* in the space $C_p^0[a,\ b]$ meaning that the above *generalized Fourier series* is a representation of f.

Theorem 3. *Let f be a piecewise smooth function on the interval $[a,\ b]$ and let $\{y_j\}_{j\in\mathbb{Z}^+}$ be the system of eigenfunctions of the SL-problem* (2)*. Then we have the following representation of the function f*

$$f_{av}(x) = \sum_{j=1}^{\infty} c_j y_j(x) \ ,$$

where the coefficients c_j are given by formula (3)*. In particular, at all points x at which f is continuous, we have*

$$f(x) = \sum_{j=1}^{\infty} c_j y_j(x) \ ,$$

Example 1. Consider the SL-problems

$$\begin{array}{ll} y''(x) + \lambda y(x) = 0 & \quad 0 < x < 1 \\ y(0) = 0 & \\ y(1) - y'(1) = 0 & \end{array}$$

It can be verified (I leave it as an exercise) that $\lambda < 0$ cannot be an eigenvalue. For $\lambda = 0$, the general solution of the ODE is $y(x) = A + Bx$. The first condition $y(0) = 0$ gives $A = 0$. For $y = Bx$, the second condition becomes is trivially satisfied ($B - B = 0$). Hence $\lambda = 0$ is an eigenvalue with eigenfunction $y_0 = x$.

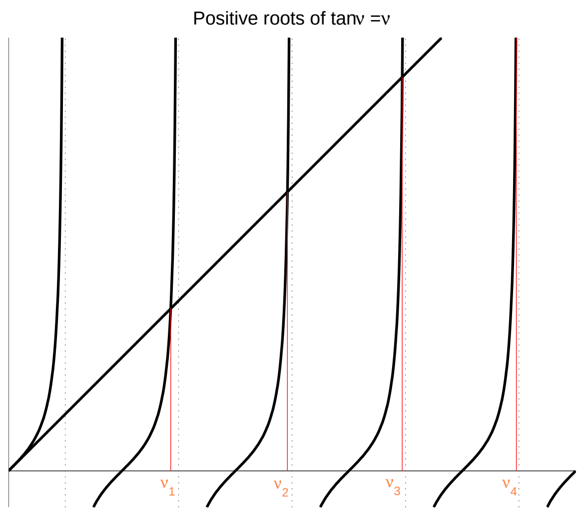
For $\lambda > 0$, we set $\lambda = \nu^2$, the general solution of the ODE is $y(x) = C_1 \cos(\nu x) + C_2 \sin(\nu x)$. The condition $y(0) = 0$ gives $C_1 = 0$. With $y = C_2 \sin(\nu x)$ the second condition $y(1) - y'(1) = 0$ gives

$$C_2 \sin(\nu) - C_2 \nu \cos(\nu) = 0 \ .$$

In order to have a nontrivial solution (so $C_2 \neq 0$), the parameter $\nu > 0$ must satisfy the equation

$$\sin(\nu) - \nu\cos(\nu) = 0 \quad \Leftrightarrow \quad \tan(\nu) = \nu \ .$$

This equation has a unique solution ν_j in each interval $(j\pi,\ j\pi + (\pi/2))$. Indeed,



consider the function $g(\nu) = \sin\nu - \nu\cos\nu$. We have $g(j\pi) = -(j\pi)(-1)^j$ and

$g(j\pi + (\pi/2)) = (-1)^j$. Thus $g(j\pi)g(j\pi + (\pi/2)) = -j\pi < 0$. The intermediate value theorem implies that g has a zero ν_j between $j\pi$ and $j\pi + (\pi/2)$. Since the derivative $g'(\nu) = \nu \sin \nu$ does not change sign in the interval $[j\pi,\ j\pi + (\pi/2)]$, then g has a unique zero in the interval. The same argument shows that g has no zeros in the interval $[j\pi + (\pi/2),\ j\pi + \pi]$.

The set of eigenvalues and eigenfunctions of the SL-problem consists therefore of $\lambda_0 = 0$, $y_0(x) = x$, and for $j \in \mathbb{Z}^+$

$$\lambda_j = \nu_j^2, \quad y_j(x) = \sin(\nu_j x), \quad \text{where } \tan \nu_j = \nu_j, \ \ \nu_j \in (j\pi,\ j\pi + (\pi/2))$$

The approximate values of the first eight values of ν_j and λ_j are listed in the following table

j	1	2	3	4	5	6	7	8
ν_j	4.493	7.725	10.904	14.066	17.220	20.371	23.519	26.666
λ_j	20.2	59.7	118.9	197.9	296.6	415.0	553.2	711.1

Example 2. Let us find the representation of the function $f(x) = 1$ on $[0,\ 1]$ in terms of the eigenfunctions

$$x, \quad \sin(\nu_j x), \quad j = 1,\ 2,\ 3,\ \cdots$$

of the SL-problem of Example 1. That is,

$$1 = c_0 x + \sum_{j=1}^{\infty} c_j \sin(\nu_j x)\ , \qquad x \in (0,\ 1)\ .$$

The coefficients are given by

$$c_0 = \frac{< 1, x >}{||x||^2}, \qquad c_j = \frac{< 1, \sin(\nu_j x)}{|| \sin(\nu_j x)||^2}, \quad j = 1,\ 2,\ 3,\ \cdots$$

Note that the weight in this example is $r(x) \equiv 1$. We have,

$$< 1, x >= \int_0^1 x dx = \frac{1}{2}$$

and

$$< 1, \sin(\nu_j x) >= \int_0^1 \sin(\nu_j x) dx = \frac{1 - \cos \nu_j}{\nu_j}\ .$$

For the norms of the eigenfunctions, we have

$$||x||^2 = \int_0^1 x^2 dx = \frac{1}{3}$$

and (recall that $\sin \nu_j = \nu_j \cos \nu_j$)

$$\begin{aligned} || \sin(\nu_j x)||^2 &= \int_0^1 \sin^2(\nu_j x) dx = \frac{1}{2} \int_0^1 (1 - \cos(2\nu_j x)) dx \\ &= \frac{1}{2} \left[x - \frac{\sin(2\nu_j x)}{2\nu_j} \right]_0^1 = \frac{1}{2} \left(1 - \frac{\sin(2\nu_j)}{2\nu_j} \right) \\ &= \frac{1}{2} \left(1 - \frac{\sin \nu_j \cos \nu_j}{\nu_j} \right) \\ &= \frac{1 - \cos^2 \nu_j}{2} = \frac{\sin^2 \nu_j}{2}\ . \end{aligned}$$

We have therefore,

$$c_0 = \frac{1/2}{1/3} = \frac{3}{2}\,, \quad \text{and} \quad c_j = \frac{(1-\cos\nu_j)/\nu_j}{(1-\cos^2\nu_j)/2} = \frac{2}{\nu_j(1+\cos\nu_j)}\,, \quad j \geq 1\ .$$

The expansion of $f(x) = 1$ is

$$1 = \frac{2}{3}x + \sum_{j=1}^{\infty} \frac{2}{\nu_j(1+\cos\nu_j)} \sin(\nu_j x) \qquad x \in (0,\ 1)\ .$$

By using the approximate values of the ν_j given in the table, we get

$$\begin{aligned} 1 \ &= \frac{2x}{3} + 0.569\sin(4.493x) + 0.229\sin(7.725x) + \\ &\quad +0.202\sin(10.904x) + 0.133\sin(14.067x) + \cdots \end{aligned}$$

4. Applications to Boundary Value Problems

We illustrate the use of SL problems in solving boundary value problems.

Example 1. Consider the problem dealing with the one-dimensional heat propagation. The temperature $u(x,t)$ satisfies the problem

$$\begin{array}{ll} u_t = k u_{xx} & 0 < x < L, \quad t > 0 \\ u(0,t) = 0, \quad u(L,t) = a u_x(L,t) & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{array}$$

where a is a positive constant. Suppose that $a < L$. Let $u(x,t) = X(x)T(t)$ be a nontrivial solution of the homogeneous part ($u_t = k u_{xx}$, $u(0,t) = 0$, and $u(L,t) = a u_x(L,t)$). The separation of variables leads to the following ODE problems for X and T:

$$\left\{ \begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X(L) = aX'(L) \end{array} \right. \qquad \text{and} \quad T'(t) + k\lambda T(t) = 0.$$

The X-problem is a regular SL-problem. It is analogous to the problem of the example of the previous section. This time however $\lambda = 0$ is not an eigenvalue. Indeed, for $\lambda = 0$, the general solution of the ODE is $X(x) = A + Bx$, the first condition $X(0) = 0$ gives $A = 0$, and the second implies that $BL = aB$. Since $a < L$, then $B = 0$ and $X \equiv 0$.

The eigenvalues and eigenfunctions are

$$\lambda_j = \nu_j^2, \quad X_j(x) = \sin(\nu_j x), \quad j \in \mathbb{Z}^+$$

where ν_j is the j-th positive root of the equation

$$\sin(\nu L) = a\nu\cos(\nu L) \quad \Leftrightarrow \quad \tan(\nu L) = a\nu\ .$$

For each j, the corresponding solution solution of the homogeneous part is $u_j(x,t) = \exp(-k\nu_j^2 t)\sin(\nu_j x)$. The superposition principle gives the general series solution as

$$u(x,t) = \sum_{j=1}^{\infty} c_j \mathrm{e}^{-k\nu_j^2 t} \sin(\nu_j x)\ .$$

In order for such a solution to satisfy the nonhomogeneous condition, we need to have

$$u(x,0) = f(x) = \sum_{j=1}^{\infty} c_j \sin(\nu_j x)\ .$$

Therefore,

$$c_j = \frac{< f(x), \sin(\nu_j x) >}{|| \sin(\nu_j x)||^2} = \frac{\int_0^L f(x)\sin(\nu_j x)dx}{\int_0^L \sin^2(\nu_j x)dx}$$

The norms (squared) of the eigenfunctions $\sin(\nu_j x)$ can be calculated as in the previous section:

$$\begin{aligned}
|| \sin(\nu_j x)||^2 &= \int_0^L \sin^2(\nu_j x)dx = \frac{1}{2}\int_0^L (1 - \cos(2\nu_j x))dx \\
&= \frac{1}{2}\left[x - \frac{\sin(2\nu_j x)}{2\nu_j}\right]_0^L = \frac{1}{2}\left(L - \frac{\sin(2\nu_j L)}{2\nu_j}\right) \\
&= \frac{1}{2}\left(L - \frac{\sin(\nu_j L)\cos(\nu_j L)}{\nu_j}\right) \\
&= \frac{L - a\cos^2(\nu_j L)}{2} .
\end{aligned}$$

If for example $f(x) = x$, then

$$\begin{aligned}
< f(x), \sin(\nu_j x) > &= \int_0^L x\sin(\nu_j x)dx = \left[\frac{-x\cos(\nu_j x)}{\nu_j}\right]_0^L + \int_0^L \frac{\cos(\nu_j x)}{\nu_j}dx \\
&= -\frac{L\cos(\nu_j L)}{\nu_j} + \frac{\sin(\nu_j L)}{\nu_j^2} \\
&= -\frac{L\cos(\nu_j L)}{\nu_j} + \frac{a\nu_j\cos(\nu_j L)}{\nu_j^2} \\
&= (a - L)\frac{\cos(\nu_j L)}{\nu_j}
\end{aligned}$$

The coefficients c_j are

$$c_j = \frac{(a-L)\dfrac{\cos(\nu_j L)}{\nu_j}}{\dfrac{L - a\cos^2(\nu_j L)}{2}} = 2(a-L)\frac{\cos(\nu_j L)}{\nu_j(L - a\cos^2(\nu_j L))} .$$

The solution of the BVP is

$$u(x,t) = (a-L)\sum_{j=1}^{\infty} \frac{\cos(\nu_j L)}{\nu_j(L - a\cos^2(\nu_j L))}\exp(-k\nu_j^2 t)\sin(\nu_j x) .$$

Example 2. The following problem models the wave propagation in a nonhomogeneous string (the mass density of the string is not constant).

$$\begin{array}{ll}
u_{tt} = (1+x)^2 u_{tt}, & 0 < x < 1, \quad t > 0, \\
u(0,t) = 0, \quad u(1,t) = 0, & t > 0, \\
u(x,0) = f(x), & 0 < x < 1, \\
u_t(x,0) = g(x), & 0 < x < 1.
\end{array}$$

The mass density of the string at the point x is $1/(1+x)^2$. The separation of variables $u(x,t) = X(x)T(t)$ for the homogeneous part leads to the following ODE problems

$$\begin{cases} X'' + \dfrac{\lambda}{(1+x)^2}X = 0 \\ X(0) = X(1) = 0 \end{cases} \quad \text{and} \quad T'' + \lambda T = 0 .$$

The X-problem is a regular SL-problem with weight $r(x) = 1/(1+x)^2$.

Now we need to find the eigenvalues and eigenfunctions of the X-problem. The ODE can be written as $(1+x)^2X'' + \lambda X = 0$. It is a Cauchy-Euler type equation in the variable $(1+x)$. We seek then solutions of the form $(1+x)^m$. It leads to the characteristic equation $m^2 - m + \lambda = 0$. The characteristic roots are therefore

$$m_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$$

Depending on the parameter λ, we consider three cases.

If $\lambda < 1/4$, then m_1 and m_2 are real and distinct. The general solution of the ODE is $X(x) = C_1(1+x)^{m_1} + C_2(1+x)^{m_2}$. The boundary conditions give

$$X(0) = C_1 + C_2 = 0, \quad X(1) = C_1 2^{m_1} + C_2 2^{m_2} = 0.$$

The only solution is $C_1 = C_2 = 0$ and such λ cannot be an eigenvalue.

If $\lambda = 1/4$, then $m_1 = m_2 = 1/2$. The general solution of the ODE is $X(x) = A\sqrt{1+x} + B\sqrt{1+x}\ln(1+x)$. In this case again the boundary conditions imply that $A = B = 0$ and $\lambda = 1/4$ cannot be an eigenvalue.

If $\lambda > 1/4$, then m_1 and m_2 are complex conjugate. Set $\lambda = \nu^2 + (1/4)$, with $\nu > 0$, so that the characteristic roots are

$$m_1 = \frac{1}{2} + i\nu, \qquad m_2 = \frac{1}{2} - i\nu \,.$$

The $\mathbb{C}$-valued independent solutions of the ODE are

$$(1+x)^{m_{1,2}} = (1+x)^{1/2}(1+x)^{\pm i\nu} = \sqrt{1+x}\mathrm{e}^{\pm i\nu\ln(1+x)} \,.$$

The general $\mathbb{R}$-valued solution is therefore

$$X(x) = A\sqrt{1+x}\cos(\nu\ln(1+x)) + B\sqrt{1+x}\sin(\nu\ln(1+x)) \,.$$

The condition $X(0) = 0$ gives $A = 0$, and then the condition $X(1) = 0$ gives $B\sqrt{2}\sin(\nu\ln 2) = 0$. In order to obtain a nontrivial solution ($B \neq 0$), the parameter ν needs to satisfy

$$\sin(\nu\ln 2) = 0 \quad \Leftrightarrow \quad \nu\ln 2 = j\pi, \quad j \in \mathbb{Z}^+ \,.$$

The eigenvalues and eigenfunctions of the X-problem are therefore

$$\lambda_j = \frac{1}{4} + \nu_j^2, \quad X_j(x) = \sqrt{1+x}\sin(\nu_j\ln(1+x)), \quad \text{where} \quad \nu_j = \frac{j\pi}{\ln 2} \,.$$

For each $j \in \mathbb{Z}^+$, the corresponding independent solutions of the T-problem are

$$T_j^1(t) = \cos(\sqrt{\lambda_j}\,t), \quad \text{and} \quad T_j^2(t) = \sin(\sqrt{\lambda_j}\,t) \,.$$

The general series solution of the homogeneous part of the problem is

$$u(x,t) = \sum_{j=1}^{\infty} \left[A_j\cos(\sqrt{\lambda_j}\,t) + B_j\sin(\sqrt{\lambda_j}\,t)\right]\sqrt{1+x}\sin(\nu_j\ln(1+x))$$

In order for such a series to satisfy the nonhomogeneous conditions, we need

$$\begin{aligned} u(x,0) &= f(x) = \sum_{j=1}^{\infty} A_j\sqrt{1+x}\sin(\nu_j\ln(1+x)) \\ u_t(x,0) &= g(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\,B_j\sqrt{1+x}\sin(\nu_j\ln(1+x)) \end{aligned}$$

The coefficients A_j and B_j are given by

$$\begin{aligned} A_j &= \frac{< f(x), \sqrt{1+x}\sin(\nu_j \ln(1+x)) >_r}{||\sqrt{1+x}\sin(\nu_j \ln(1+x))||_r^2}, \\ B_j &= \frac{< g(x), \sqrt{1+x}\sin(\nu_j \ln(1+x)) >_r}{\sqrt{\lambda_j}||\sqrt{1+x}\sin(\nu_j \ln(1+x))||_r^2} \end{aligned}$$

5. Other Sturm-Liouville Problems

Other SL-problems important in applications include the *periodic Sturm-Liouville problem*:

$$\begin{aligned} &(p(x)y')' + [\lambda r(x) - q(x)]\, y = 0 \\ &y(a) = y(b) \\ &y'(a) = y'(b) \end{aligned}$$

where the coefficient p satisfies $p(a) = p(b)$. The set of eigenvalues is as in Theorem 1 but this time for each eigenvalue λ_j, there are two independent eigenfunctions $y_j^1(x)$ and $y_j^2(x)$.

Example 1. The following periodic SL-problem was encountered in connection with BVPs in polar coordinates

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \qquad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

The eigenvalues are $\lambda_j = j^2$ and the eigenfunctions are $\Theta_j^1(\theta) = \cos(j\theta)$ and $\Theta_j^2(\theta) = \sin(j\theta)$.

The hypotheses on the coefficients of the ODE in a regular SL-problem includes $p(x) > 0$ and $r(x) > 0$ on $[a,\ b]$. If any one these hypotheses is weakened, it lead to the *singular Sturm-Liouville problem.* For some singular problems, the boundary conditions can also be weakened or are not needed. The following singular SL-problem will be studied in more details. It is related to *Bessel functions.*

$$\begin{aligned} &(xy')' + \left(-\frac{m^2}{x} + \lambda x\right) y = 0, \quad 0 < x < L, \\ &y(L) = 0 \end{aligned}$$

Here $p(x) = x$ and $r(x) = x$, both functions vanish at $x = 0$. The boundary condition at $x = 0$ is not needed. This equation is related to boundary value problems in cylindrical coordinates.

Another singular problem of interest is the *Legendre equation*

$$\big((1-x^2)y'\big)' + n(n-1)y = 0 \qquad -1 < x < 1$$

This time $r(x) = 1$ and $p(x) = 1 - x^2$ vanishes at both ends ± 1 of the interval. No boundary conditions are needed. This problem appears in connection to boundary value problems in spherical coordinates.

6. Exercises

For exercises 1 to 4: (a) find the eigenvalues and eigenfunctions of the Sturm-Liouville problems; (b) find the generalized Fourier series of the functions $f(x) = 1$ and $g(x) = x$.

Exercise 1. $y'' + \lambda y = 0,\ \ 0 < x < 1,\ \ y(0) = 0$ and $y'(1) = 0$

Exercise 2. $y'' + \lambda y = 0, \ -1 < x < 1, \ y(-1) = y(1)$ and $y'(-1) = y'(1)$ (periodic SL problem)

Exercise 3. $y'' + \lambda y = 0, \ 0 < x < 1, \ y(0) = 0$ and $y(1) + 2y'(1) = 0$

Exercise 4. $y'' + \lambda y = 0, \ 0 < x < 1, \ y(0) = y'(0)$ and $y(1) = y'(1)$

Exercise 5. Consider the problem

$$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < L, \quad y(1) = 0, \quad y(L) = 0,$$

with $L > 1$.

(1) Put the ODE in adjoint form: $(py')' + (q + \lambda r)y = 0$ (*Hint*: multiply by $1/x$).

(2) What is the inner product related to this problem?

(3) Find the eigenvalues and eigenfunctions (note: the ODE is Cauchy-Euler).

(4) Find the generalized Fourier series of the function $f(x) = 1$ (*Hint*: when computing the Fourier coefficients c_j, you can use the substitution $t = \ln x$ in the integral).

(5) Same question for the function $g(x) = x$.

Exercise 6. Same questions as in Exercise 5 for the SL-problem

$$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < L, \quad y'(1) = 0, \quad y'(L) = 0,$$

Exercise 7. Solve the BVP

$$\begin{array}{ll} u_t = 2u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0 & t > 0 \\ 2u(\pi,t) + u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin x & 0 < x < \pi \end{array}$$

Exercise 8. Solve the BVP

$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = 0 & t > 0 \\ u(\pi,t) = u_x(\pi,t) & t > 0 \\ u(x,0) = 1 & 0 < x < \pi \end{array}$$

Exercise 9. Solve the BVP

$$\begin{array}{ll} u_{tt} = c^2u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) - u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin x & 0 < x < \pi \\ u_t(x,0) = 0 & 0 < x < \pi \end{array}$$

Exercise 10. Solve the BVP

$$\begin{array}{ll} u_{tt} = c^2u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) - u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = f(x) & 0 < x < \pi \end{array}$$

where

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < (\pi/2), \\ 1 & \text{if } (\pi/2) < x < \pi. \end{cases}$$

Exercise 12. Solve the BVP

$$\begin{array}{ll} u_{tt} = u_{xx} & 0 < x < \pi, \quad t > 0, \\ u_x(0,t) = au(0,t) & t > 0 \\ u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \\ u_t(x,0) = 1 & 0 < x < \pi \end{array}$$

Exercise 13. Solve the BVP

$$\begin{array}{ll} u_t = (1+x)^2 u_{xx} & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0 \quad u(1,t) = 0 & t > 0 \\ u(x,0) = x(1-x)\sqrt{1+x} & 0 < x < 1 \end{array}$$