Fall 2022 - Real Analysis
Homework 1

1. For a nonempty set of real numbers E, show that $\inf E = \sup E$ if and only if E consists of a single point.

2. Show that the irrational numbers are dense in $\mathbb{R}$.

3. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

4. Show that if $r > 0$, then, for each natural number n, $(1+r)^n \geq 1 + nr$.

5. Show that the set $\mathbb{Z}$ of integers is countable.

6. Show that a set A is countable if and only if there is a one-to-one mapping of A to $\mathbb{N}$.

7. Let both $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be one-to-one and onto. Show that the composition $g \circ f : A \longrightarrow B$ and the inverse $f^{-1} : B \longrightarrow A$ are also one-to-one and onto.

8. Show that $2^{\mathbb{N}}$, the collection of all sets of natural numbers, is uncountable. Show that $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, is not countable.

9. Show that the Cartesian product of a finite collection of countable sets is countable.

10. Show that the intervals $[0,\ 1)$ and $(0,\ 1)$ are equipotent. Show that any two nondegenerate intervals of real numbers are equipotent.

11. Show that $(0,\ 1)^2$ is equipotent to $(0,1)$. Deduce that $\mathbb{R} \times \mathbb{R}$ is equipotent to $\mathbb{R}$.

12. Is the set of rational numbers open or closed?

13. What are the sets of real numbers that are both open and closed?

14. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

15. A point x is called an accumulation point of a set E provided it is a point of closure of $E \backslash \{x\}$.

(1) Show that the set E' of accumulation points of E is a closed set.
(2) Show that $\overline{E} = E \cup E'$.

16. A point x is called an isolated point of a set E provided there is an $r > 0$ for which $(x-r,\ x+r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.

17.A point x is called an interior point of a set E if there is an $r > 0$ such that the open interval $(x-r,\ x+r)$ is contained in E. The set of interior points of E is called the interior of E denoted by $\mathrm{int}(E)$. Show that

(1) E is open if and only if $E = \mathrm{int}(E)$.
(2) E is dense if and only if $\mathrm{int}(\mathbb{R} \backslash E) = \emptyset$.

18. Show that the Nested Set Theorem is false if F_l is unbounded.

19. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.

20. Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

21. Show that the collection of Borel sets is the smallest σ-algebra that contains intervals of the form $[a,\ b)$, where $a < b$.

22. Show that each open set is an F_σ set.