Fall 2022 - Real Analysis
Homework 10

1. Let C be a countable subset of the nondegenerate closed, bounded interval $[a,\ b]$. Show that there is an increasing function on $[a,\ b]$ that is continuous only at points in $[a,\ b]\backslash C$.

2. Show that there is a strictly increasing function on $[0,\ 1]$ that is continuous only at the irrational numbers in $[0,\ 1]$.

3. Let f be a monotone function on a subset $E \subset R$. Show that f is continuous except possibly at a countable number of points in E.

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x \sin \dfrac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. Find $D^+f(0)$ and $D^-f(0)$.

5. (a) Assume that $f : [a,\ b] \longrightarrow \mathbb{R}$ is continuous and that there exists $\epsilon > 0$ such that $D^-f(x) \geq \epsilon$ for all $x \in (a,\ b)$. Prove that f is monotone increasing on $[a,\ b]$. Here $D^-f(x) = \liminf\limits_{h\to 0^-} \dfrac{f(x+h)-f(x)}{h}$. *Hint*: Use a proof by contradiction. Suppose that f is not increasing and deduce that there exists $y \in (a,\ b)$ such that $D^-f(y) \leq 0$.

(b) Assume that $f : [a,\ b] \longrightarrow \mathbb{R}$ is continuous that $D^-f(x) \geq 0$ for all $x \in (a,\ b)$. Prove that f is monotone increasing on $[a,\ b]$. *Hint:* Consider the function $g(x) = f(x) + \epsilon x$ where $\epsilon > 0$.

6. Let f be continuous on $\mathbb{R}$. Is there an open interval on which f is monotone? *Hint*: The Weierstrass function $W(x)$ given by

$$W(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x) \quad \text{with } b \in (0,1), \quad a \in \mathbb{N}\,, \text{ such that } ab > 1 + \frac{3\pi}{2}\,,$$

is continuous in $\mathbb{R}$ and is not differentiable at any point of $\mathbb{R}$.

7. Let I and J be closed, bounded intervals and $\gamma > 0$ be such that $\ell(I) > \gamma\,\ell(J)$. Assume that $I \cap J \neq \emptyset$. Show that if $\gamma \geq 1/2$, then $J \subset 5 * I$, where $5 * I$ denotes the interval with the same center as I and five times its length. Is the same true if $0 < \gamma < 1/2$?

8. Show that a set E of real numbers has measure zero if and only if there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ for which each point in E belongs to infinitely many of the I_k's and $\sum_{k=1}^{\infty} \ell(I_k) < \infty$.

9. (Riesz-Nagy) Let E be a set of measure zero contained in the open interval $(a,\ b)$. According to the preceding problem, there is a countable collection of open intervals contained in (a, b), $\{(c_k,\ d_k)\}_{k=1}^{\infty}$ for which each point in E belongs to infinitely many intervals' in the collection and $\sum_{k=1}^{\infty}(d_k - c_k) < \infty$. Define a function f on $(a,\ b)$ by

$$f(x) = \sum_{k=1}^{\infty} \ell((c_k,\ d_k) \cap (-\infty,\ x)) \quad \text{for all } x \in (a,\ b)$$

Show that f is increasing and fails to be differentiable at each point in E. *Hint*: Note that if $x \in E$, then there exists $\{(c_{k_j},\ d_{k_j})\}_{j=1}^{\infty}$ such that $x \in (c_{k_j},\ d_{k_j})$ for all j. Use this to show that for every $n \in \mathbb{N}$ there exists $\delta > 0$ such that $f(x+h) - f(x) \geq nh$ for all $0 < h < \delta$.

10. Compute the upper and lower derivatives of the characteristic function of the rationals. $\chi_{\mathbb{Q}}$.

11. Let g be integrable over $[a,\ b]$. Define the antiderivative of g to be the function f defined on $[a,\ b]$ by $f(x) = \displaystyle\int_a^x g(t)dt$ for $x \in [a,\ b]$. Show that f is differentiable almost everywhere on $(a,\ b)$. *Hint:* Start with the case g nonnegative.

12. Show that a strictly increasing function that is defined on an interval is measurable and then use this to show that a monotone function that is defined on an interval is measurable

13. The goal of this exercise is to show that a continuous function f on $[a,\ b]$ is lipschitz if its upper and lower derivatives are bounded on $(a,\ b)$ (i.e. there exists $M > 0$ such $|D^{\pm}f(x)| < M$ for all $x \in (a, b)$)

- Let h be a function defined on an interval $(\alpha,\ \beta)$. Show that if c is a local minimum of h, then $D^-h(c) \leq 0 \leq D^+h(c)$.

- Let $a < x_1 < x_2 < b$. Consider the function
$$g(x) = f(x) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}\,(x - x_1)\,.$$
Show that there exists $c \in [x_1,\ x_2]$ such that $D^- g(c) \le 0 \le D^+ g(c)$.
- Deduce that $|f(x_2) - f(x_1)| \le M(x_2 - x_1)$

14. Let $f(x) = \sin x$ on $[0,\ 2\pi]$. Find two increasing functions g and h for which $f = g - h$ on $[0,\ 2\pi]$.

15. Determine whether or not the functions f and g defined below are in BV$[-1,\ 1]$:
$$f(x) = \begin{cases} x^2 \cos \dfrac{1}{x^2} & \text{if } 0 < |x| \le 1 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \cos \dfrac{1}{x} & \text{if } 0 < |x| \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

16. Let $\{f_n\}$ be a sequence of real-valued functions on $[a,\ b]$ that converges pointwise on $[a,\ b]$ to the real-valued function f. Show that $V(f) \le \liminf_{n\to\infty} V(f_n)$.

17. Let f be a function that fails to be of bounded variation on $[0,\ 1]$. Show that there is a point $x_0 \in [0,\ 1]$ such that f fails to be of bounded variation on each nondegenerate closed subinterval of $[0,\ 1]$ that contains x_0.