Fall 2022 - Real Analysis
Homework 11

1. Show that if f is an increasing function on $[a,\ b]$, then f is absolutely continuous on $[a,\ b]$ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for any measurable subset $E \subset [a,\ b]$, with $m(E) < \delta$, we have $m(f(E)) < \epsilon$.

2. Use the preceding problem to show that an increasing absolutely continuous function f on $[a,\ b]$ maps sets of measure zero onto sets of measure zero. Conclude that the Cantor-Lebesgue function ϕ is not absolutely continuous on $[0,\ 1]$ since the function ψ, defined by $\psi(x) = x + \phi(x)$ for $0 \le x \le 1$, maps the Cantor set to a set of measure 1.

3. Let f be an increasing absolutely continuous function on $[a,\ b]$. Show that f maps measurable sets to measurable sets. *Hint:* First show that f maps an F_σ to an F_σ-set and a set of measure zero to a set of measure zero (previous problem).

4. Show that both the sum and product of absolutely continuous functions are absolutely continuous.

5. Let f and g be functions on $[-1,\ 1]$ given by $f(x) = x^{1/3}$ and $g(x) = x^2 \cos(\pi/x)$ if $x \neq 0$ and $g(0) = 0$.

(1) Show that $f, g \in \mathrm{AC}[-1,\ 1]$.

(2) Find $V(f \circ g, P)$, where P is the partition of $[-1,\ 1]$ that consists of the points

$$x_0 = -1,\ x_1 = 0,\ x_2 = \frac{1}{2n},\ x_3 = \frac{1}{2n-1}, \cdots,\ x_{2n} = \frac{1}{2},\ x_{2n+1} = 1.$$

(3) Deduce that $f \circ g \notin \mathrm{BV}[-1,\ 1]$ and so $f \circ g \notin \mathrm{AC}[-1,\ 1]$.

6. Let f be Lipschitz on $\mathbb{R}$ and g be absolutely continuous on $[a,\ b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a,\ b]$.

7. Let f be absolutely continuous on $\mathbb{R}$ and g be absolutely continuous and strictly monotone on $[a,\ b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a,\ b]$.

8. Let f be continuous on $[a,\ b]$ and differentiable almost everywhere on $(a,\ b)$. Show that

$$\int_a^b f'(x)dx = f(b) - f(a) \iff \int_a^b \lim_{n\to\infty} D_{1/n} f(x)dx = \lim_{n\to\infty} \int_a^b D_{1/n} f(x)dx$$

9. Let f be continuous on $[a,\ b]$ and differentiable almost everywhere on $(a,\ b)$. Show that if $\{D_{1/n} f\}_n$ is uniformly integrable, then $\int_a^b f'(x)dx = f(b) - f(a)$. *Hint:* Vitali Convergence Theorem.

10. Let f be continuous on $[a,\ b]$ and differentiable almost everywhere on $(a,\ b)$. Suppose that there exists a nonnegative function $g \in \mathcal{L}(a,\ b)$ such that $\left|D_{1/n} f\right| \le g$ a.e. on $[a,\ b]$ for all $n \in \mathbb{N}$. Show that $\int_a^b f'(x)dx = f(b) - f(a)$.

11. (Integration by Part). Let $f, g \in \mathrm{AC}[a,\ b]$. Show that

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$

12. Let $f \in \mathrm{AC}[a,\ b]$. Show that f is Lipschitz on $[a,\ b]$ if and only if there is a $c > 0$ for which $|f'| \le c$ a.e.on $[a,\ b]$.

13. Let $f \in \mathrm{AC}[a,\ b]$ be **strictly increasing**. Show that

(1) For any open set $U \subset [a,\ b]$, $\int_U f'(x)dx = m(f(U))$.

(2) For any G_δ set $G \subset [a,\ b]$, $\int_G f'(x)dx = m(f(G))$ *Hint:* You might need to use the property of AC continuous function g $\forall \epsilon > 0,\ \exists \delta > 0$ such that if $m(E) < \delta$, then $m(g(E)) < \epsilon$.

(3) For any set $Z \subset [a,\ b]$ with measure 0, $\int_Z f'(x)dx = m(f(Z)) = 0$.

(4) For any measurable set $E \subset [a,\ b]$, $\int_E f'(x)dx = m(f(E))$.

(5) Let $c = f(a)$ and $d = f(b)$. Let ϕ be a simple function on $[c,\ d]$. Show that

$$\int_c^d \phi(y)dy = \int_a^b \phi(f(x))f'(x)dx.$$

(6) Show that if $g \in \mathcal{L}(c,\ d)$ is nonnegative, then $\int_c^d g(y)dy = \int_a^b g(f(x))f'(x)dx$.