Fall 2022 - Real Analysis
Homework 12

1. Show that the collection of all simple functions on E is dense in $L^p(E)$.

2. Let $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$.

- For $t \geq 0$, let $A_t = \{x \in \mathbb{R}^n : \ |f(x)|^p > t\}$ and consider the function $g(t,x)$ defined on $[0,\ \infty) \times \mathbb{R}^n$ by $g(t,x) = \chi_{A_t}(x)$. Show that
$$\int_E |f(x)|^p dx = \int_{[0,\ \infty)\times\mathbb{R}^n} g(t,x) dt dx.$$
- Show that $\displaystyle\int_0^\infty m(A_t)dt = \int_0^\infty pt^{p-1} m\left(\{x \in \mathbb{R}^n : |f(x)| > t\}\right) dt.$
- Deduce that $\displaystyle\int_E |f(x)|^p dx = \int_0^\infty pt^{p-1} m\left(\{x \in \mathbb{R}^n : |f(x)| > t\}\right) dt.$

Hint: You can use Fubini's Theorem.

3. When does Hölder's inequality becomes an equality? When does equality Minkowski inequality becomes an equality?

4. Let $1 \leq r < p < s$. Show that $L^r(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. *Hint:* For $f \in L^p(\mathbb{R}^n)$, consider the sets $E_1 = \{x \in \mathbb{R}^n : \ |f(x)| < 1\}$ and $E_2 = \{x \in \mathbb{R}^n : \ |f(x)| \geq 1\}$.

5. Prove that if (p,q) is a conjugate pair with $p \geq 1$, $\{f_j\}_j \in L^p(\mathbb{R}^n)$ converges to f in L^p, and $g \in L^q(\mathbb{R}^n)$, then $\displaystyle\lim_{j\to\infty} \int_{\mathbb{R}^n} f_j g dx = \int_{\mathbb{R}^n} f g dx$

6. Consider the sequence of functions $\{g_n\}_n$ on $[0,\ 1]$ given by $g_n = n\chi_{A_n}$ where $A_n = [0,\ 1/n^3]$.

- Show that if $f \in L^2[0,\ 1]$, then $\displaystyle\lim_{n\to\infty} \int_0^1 f(x) g_n(x) dx = 0$.
- Find a function $f \in L^1[0,\ 1]$ such that $\displaystyle\lim_{n\to\infty} \int_0^1 f(x) g_n(x) dx \neq 0$. *Hint:* Consider $f(x) = x^\alpha$ for some real number α.

7. Let $1 \leq p \leq \infty$.

- Find all $a \in \mathbb{R}$ such that $f_a(x) = x^a \chi_{[0,\ 1]}(x) \in L^p(\mathbb{R})$
- Find all $b \in \mathbb{R}$ such that $g_b(x) = x^b \chi_{[1,\ \infty)}(x) \in L^p(\mathbb{R})$

8. (L^p-version of Chebychev's inequality). Let $E \subset \mathbb{R}^n$ and $f : E \longrightarrow \overline{\mathbb{R}}$ be a measurable function. Prove that for any $\lambda > 0$, we have
$$m\left(\{|f| > \lambda\}\right) \leq \frac{1}{\lambda^p} \int_E |f|^p dx.$$

9. Let $E \subset \mathbb{R}^n$ with $m(E) < \infty$ and let $f : E \longrightarrow \overline{\mathbb{R}}$ be such that f is measurable. The aim of this exercise is to prove that $\displaystyle\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

- Case $||f||_\infty = \infty$. Show that in this case $||f||_p = \infty$ for all $p \geq 1$. *Hint:* You can use Exercise 8.
- Case $||f||_\infty = 0$. Show that in this case $||f||_p = 0$ for all p.
- Case $||f||_\infty = C$ with $0 < C < \infty$.
  (1) Show that $||f||_p^p \leq C^p\, m(E)$ and deduce that $\displaystyle\limsup_{p\to\infty} ||f||_p \leq C$.
  (2) Let $\epsilon > 0$ and $A_\epsilon = \{x \in E,\ |f(x)| \geq C - \epsilon\}$. Show that
  $$||f||_p^p \geq (C-\epsilon)^p\ m(A_\epsilon)$$
  and deduce that $\displaystyle\liminf_{p\to\infty} ||f||_p \geq C - \epsilon$
  (3) Deduce that $\displaystyle\lim_{p\to\infty} ||f||_p = C$.

10. The aim of this problem is to establish a generalized of Hölder inequality. More precisely, given a measurable set $E \subset \mathbb{R}^k$, positive numbers $p_1, \cdots, p_n$, r such that
$$\frac{1}{r} = \frac{1}{p_1} + \cdots \frac{1}{p_n},$$

and functions $f_1 \in L^{p_1}(E), \cdots f_n \in L^{p_n}(E)$, then

$$f = f_1 f_2 \cdots f_n \in L^r(E) \quad \text{and} \quad ||f||_r \leq ||f_1||_{p_1} \cdot ||f_2||_{p_2} \cdots ||f_n||_{p_n}$$

- Let p, q and r be positive numbers such that $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$. Show that

$$\frac{t}{r} \leq \frac{t^{p/r}}{p} + \frac{1}{q} \qquad \forall t \geq 0\,.$$

- Show that

$$\frac{(ab)^r}{r} \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \forall\, a, b \geq 0\,.$$

  Hint: Use the previous inequality with $t = a^r b^{r-q}$.
- Let $f \in L^p(E),\ g \in L^q(E)$ and r such that $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$. Show that $fg \in L^r(E)$ and $||fg||_r \leq ||f||_p\, ||g||_q$.

  Hint: Let $F = \dfrac{f}{||f||_p}$ and $G = \dfrac{g}{||g||_p}$. Use the previous inequality with $a = F$ and $b = G$.
- Use induction to establish the generalized Hölder inequality.

11. Let $p, q \in (0,\ \infty]$ with $p < q$ and let E be a measurable subset of $\mathbb{R}^n$.

- Show that if $m(E) < \infty$, then $L^q(E) \subsetneq L^p(E)$ and

$$||f||_p \leq m(E)^{\frac{1}{p}-\frac{1}{q}}\, ||f||_q \qquad \forall f \in L^q(E)$$

  Hint: Apply the generalized Hölder inequality to f and χ_E.
- Show that if $E = \mathbb{R}^n$, then $L^q(E) \not\subseteq L^p(E)$ and $L^p(E) \not\subseteq L^q(E)$. *Hint*: Consider $f(x) = |x|^\alpha$.

12. The aim of this problem is to show (by contradiction) that for a measurable function $\phi : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ and for $1 \leq p < \infty$, if $f\phi \in L^p(\mathbb{R}^n)$ for all $f \in L^p(\mathbb{R}^n)$, then $\phi \in L^\infty(\mathbb{R}^n)$.

- Suppose that $\phi \notin L^\in(\mathbb{R}^n)$. Show that for there exists a strictly increasing sequence $\{k_j\}_{j=1}^\infty$ in $\mathbb{N}$, such that for every $j \in \mathbb{N}$, the set $E_j = \{x \in \mathbb{R}^n;\ k_j \leq |\phi(x)| < k_{j+1}\}$ has a positive measure.
- Define a function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ by $f = \sum_{j=1}^{\infty} c_j \chi_{E_j}$ where the c_j's are positive numbers. Show that

$$\int_{\mathbb{R}^n} |f\phi|^p dx \geq \sum_{j=1}^{\infty} k_j^p c_j^p \mathrm{m}(E_j)\,.$$

- Show that for an appropriate choice the coefficients c_j, we have $f \in L^p(\mathbb{R}^n)$ and $f\phi \notin L^p(\mathbb{R}^n)$.