Fall 2022 - Real Analysis
Homework 13

1. Let $C^0(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ and $C_c^0(\mathbb{R}^n)$ be the space of continuous functions with compact support. That is,

$$C_c^0(\mathbb{R}^n) = \left\{ f \in C^0(\mathbb{R}^n), \quad \exists R > 0, \quad f(x) = 0 \ \ \forall x, \quad |x| > R \right\} .$$

The goal of this problem is to show that $C_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for a given $1 \leq p < \infty$. For this we will use the following Theorem known as Urysohn's Lemma (it holds in more general topological spaces but for this problem, the $\mathbb{R}^n$-version is enough).

Urysohn's Lemma: *Let A and B be closed and disjoint subests of $\mathbb{R}^n$. Then there exists a continuous function $f : \mathbb{R}^n \longrightarrow [0,\ 1]$ such that*

$$f(x) = 0 \ \ \forall x \in A \quad \textit{and} \quad f(x) = 1 \ \ \forall x \in B .$$

- Let $E \subset \mathbb{R}^n$ be bounded and measurable. Show that for every $\epsilon > 0$, there exists $f \in C_c^0(\mathbb{R}^n)$ such that $\|f - \chi_E\|_p < \epsilon$. *Hint*: Since E is measurable then for every $\eta > 0$, there exists $F \subset E \subset U$, with F a closed set, U an open set, and $m(U \backslash F) < \eta$.
- Let $E \subset \mathbb{R}^n$ be measurable with finite measure. Show that for every $\epsilon > 0$, there exists $f \in C_c^0(\mathbb{R}^n)$ such that $\|f - \chi_E\|_p < \epsilon$.
- Let ϕ be a simple function in $L^p(\mathbb{R}^n)$. Show that for every $\epsilon > 0$, there exists $f \in C_c^0(\mathbb{R}^n)$ such that $\|f - \phi\|_p < \epsilon$.
- Show $C_c^0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

2. Let $E \subset \mathbb{R}^n$ be measurable and $1 \leq p < q < \infty$. For every $j \in \mathbb{N}$, let $f_j \in L^p(E) \cap L^q(E)$. Suppose that $f_j \longrightarrow f$ in $L^p(E)$ and $f_j \longrightarrow g$ in $L^q(E)$. Show that $f = g$ a.e. in E. *Hint*: Consider first the case E with finite measure.

3. Let $a \in \mathbb{R}$. Define the operator T_a acting on functions defined on $\mathbb{R}^n$ by $T_a f(x) = f(x - a)$. Let $1 \leq p < \infty$. Show that

$$\lim_{a \to 0} ||T_a f - f||_p = 0 \qquad \forall f \in L^p(\mathbb{R}^n) .$$

Hint: You can use the density of continuous functions in L^p.

4. Let $[a,\ b]$ be a closed and bounded interval in $\mathbb{R}$ and let P the space of all polynomials of one variable $x \in \mathbb{R}$. Show that P is dense in $L^p([a,\ b])$ for $1 \leq p < \infty$. Is P dense in $L^\infty([a,\ b])$? *Hint*: You can use the following result:

Stone-Weierstrass Theorem: Let $f \in C^0([a,\ b])$. Then there exists a sequence of polynomials $\{p_n\}_n$ of one variable in $\mathbb{R}$ such that $p_n \longrightarrow f$ uniformly on $[a,\ b]$.

5. Let $E \subset \mathbb{R}^n$ be measurable and $1 \leq p < \infty$. A series $\displaystyle\sum_{j=1}^{\infty} f_j$ is said to be absolutely convergent in $L^p(E)$ if $f_j \in L^p(E)$ for all j and $\displaystyle\sum_{j=1}^{\infty} ||f_j||_p < \infty$. Suppose $\displaystyle\sum_{j=1}^{\infty} f_j$ is absolutely convergent in $L^p(E)$. Prove the following.

- The series $\displaystyle\sum_{j=1}^{\infty} f_j(x)$ converges a.e. in E to a function f;
- $f \in L^p(E)$;
- The series $f = \displaystyle\sum_{j=1}^{\infty} f_j$ converges in L^p. That is

$$\lim_{r \to \infty} \left\| f - \sum_{j=1}^{r} f_j \right\|_p = 0 .$$

- Show that for $p = 1$ we also have

$$\int_E \sum_{j=1}^{\infty} f_j \, dx = \sum_{j=1}^{\infty} \int_E f_j \, dx .$$

6. Let $E \subset \mathbb{R}^n$ be measurable and $1 \leq p < q < \infty$. Prove the followings.

- The function $\|.\| : X = L^p(E) \cap L^q(E) \longrightarrow [0, \infty)$ given by $\|f\| = \|f\|_p + \|f\|_q$ is a norm and that $(X, \|.\|)$ is a Banach space.
- If $1 \le p < r < q < \infty$, then $L^p(E) \cap L^q(E) \subset L^r(E)$ and if σ is such that $\dfrac{1}{r} = \dfrac{\sigma}{p} + \dfrac{1-\sigma}{q}$, then

$$\|f\|_r \le \|f\|_p^\sigma \, \|f\|_q^{1-\sigma} \qquad \forall f \in L^p(E) \cap L^q(E)\,.$$

7. Let $1 \le p \le \infty$. Suppose a function $f \in L^p(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} f \phi \, dx = 0 \qquad \forall \phi \in C_c^0(\mathbb{R})\,,$$

where $C_c^0(\mathbb{R})$ is the space of continuous functions with compact support. Prove that $f = 0$ a.e. in $\mathbb{R}$.