Fall 2022 - Real Analysis
Homework 2

1. We call an extended real number $a \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ a cluster point of a sequence $\{a_n\}$ if a subsequence converges to this extended real number a. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

2. Show that a sequence $\{a_n\}$ is convergent to an extended real number if and only if there is exactly one extended real number that is a cluster point of the sequence.

3. Show that $\liminf\{a_n\} \leq \limsup\{a_n\}$.

4. Prove that if, for all n, $a_n > 0$ and $b_n \geq 0$, then $\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n)$, provided the product on the right is not of the form $0 \cdot \infty$.

5. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

6. Let p be a natural number greater than 1, and x a real number, $0 < x < 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that $x = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ and that this sequence is unique except when x is of the form $\dfrac{q}{p^n}$ for some $q \in \mathbb{N}$ and $q < p^n$, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$ for all n, the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ converges to a real number x with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the decimal expansion of x. For $p = 2$ it is called the binary expansion; and for $p = 3$, the ternary expansion.

7. Let E be a closed set of real numbers and f a real-valued function that is defined and continuous on E. Show that there is a function g defined and continuous on all of $\mathbb{R}$ such that $f(x) = g(x)$ for each $x \in E$. (*Hint: Take g to be linear on each of the intervals of which $\mathbb{R}\backslash E$ is composed.*)

8. Define the real-valued function f on $\mathbb{R}$ by setting $f(x) = \begin{cases} x & \text{if } x \notin \mathbb{Q} \\ p \sin \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms} \end{cases}$

At what points is f continuous?

Hints. You can use the followings

(1) You can start by proving that if $x \in \mathbb{R}^+\backslash\mathbb{Q}$, then for every $A > 0$, there exists $\delta > 0$ such that for every rational number $r = p/q$ (with $p, q \in \mathbb{Z}^+$) we have $q > A$.

(2) $|\sin x| < x \quad \forall x \in \mathbb{R}$

(3) $|\sin x - x| < \dfrac{x^3}{6} \quad \forall x \in \mathbb{R}$

9. Let f and g be continuous real-valued functions with a common domain E.

(1) Show that the sum, $f + g$, and product, fg, are also continuous functions.

(2) If h is a continuous function with image contained in E, show that the composition $f \circ h$ is continuous.

(3) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.

(4) Show that $|f|$ is continuous.

10. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

11. A continuous function ϕ on $[a,\ b]$ is called piecewise linear provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,\ b]$ for which ϕ is linear on each interval $[x_i,\ x_{i+1}]$. Let f be a continuous function on $[a,\ b]$ and ϵ a positive number. Show that there is a piecewise linear function ϕ on $[a,\ b]$ with $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a,\ b]$.

12. Show that a nonempty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

13. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.

14. Show that a nonempty set E of real numbers is an interval if and only if every continuous real-valued function on E has an interval as its image.

15. Show that a monotone function on an open interval is continuous if and only if its image is an interval.

16. Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.

17. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.

18. Let f be a continuous real-valued function on $\mathbb{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.

19. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f if given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E. Prove that if $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.