Fall 2022 - Real Analysis
Homework 3

1. Let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0,\ \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$

(1) Prove that if A and B are two sets in $\mathcal{A}$ with $A \subset B$, then $m(A) \leq m(B)$. This property is called monotonicity.

(2) Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

(3) Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k)$.

2. A set function c, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the counting measure.

3. Let A be the set of irrational numbers in the interval $[0, 1)$. Prove that $m^*(A) = 1$.

4. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which $E \subset G$ and $m^*(G) = m^*(E)$.

5. Let B be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^{n}$ be a finite collection of open intervals that covers B. Prove that $\sum_{j=1}^{n} m^*(I_j) \geq 1$.

6. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

7. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

8. Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form $(a,\ \infty)$, then it contains all intervals.

9. Show that every interval is a Borel set.

10. Show that:

(1) The translate of an F_σ set is also F_σ

(2) The translate of a G_δ set is also G_δ

(3) The translate of a set of measure zero also has measure zero.

11. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

12. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

13. Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set U for which $F \subset E \subset U$ and $m^*(U \backslash F) < \epsilon$.

14. Let E have finite measure. Show that there is an F_σ set F and a G_δ set G such that $F \subset E \subset G$ and $m^*(F) = m^*(E) = m^*(G)$.

15. Let E have finite outer measure. Show that if E is not measurable, then there is an open set U containing E that has finite outer measure and for which $m^*(U \backslash E) > m^*(U) - m^*(E)$.
Hint. You can use the fact that a set A is measurable if and only if for every $\epsilon > 0$, there exists an open set $U \supset A$ such that $m^*(U \backslash A) < \epsilon$.

16. (Lebesgue) Let $E \subset \mathbb{R}$ have finite outer measure. Show that E is measurable if and only if for each open, bounded interval $(a,\ b)$, $\ b - a = m^*((a,\ b) \cap E) + m^*((a,\ b) \cap E^c)$.