Fall 2022 - Real Analysis
Homework 4

1. (a) Show that rational equivalence defines an equivalence relation on any subset of $\mathbb{R}$.
(b) Explicitly find a choice set for the rational equivalence relation on $\mathbb{Q}$.
(c) Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on $\mathbb{R}$? Is this an equivalence relation on $\mathbb{Q}$?

2. Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

3. Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which $m^*(E) = m^*(G)$ while $m^*(G \backslash E) > 0$.

4. Show that there is a continuous, strictly increasing function on the interval $[0,\ 1]$ that maps a set of positive measure onto a set of measure zero.

5. Let f be an increasing function on the open interval I. For $x_0 \in I$ show that f is continuous at x_0 if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n, $a_n < x_0 < b_n$, and $\lim_{n\to\infty}(f(b_n) - f(a_n)) = 0$.

6. Show that if f is any increasing function on $[0,\ 1]$ that agrees with the Cantor-Lebesgue function ϕ on the complement of the Cantor set, then $f = \phi$ on all of $(0,\ 1)$.

7. Let f be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

8. Let the function $f : [a,\ b] \longrightarrow \mathbb{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a,\ b]$, $|f(u) - f(v)| \leq c\,|u - v|$. Show that f maps a set of measure zero onto a set of measure zero. Show that f maps an F_σ set onto an F_σ set. Conclude that f maps a measurable set to a measurable set.

9. Let F be the subset of $[0,\ 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0,\ 1] \backslash F$ dense in $[0,\ 1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.

10. Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (*Hint: Consider the complement of the generalized Cantor set of the preceding problem.*)

11. A nonempty subset $X \subset \mathbb{R}$ is called perfect if it has no isolated points (or equivalently if it is closed and each open interval containing any point in X contains infinitely many points of X). Show that the Cantor set is perfect. (*Hint: The endpoints of all of the subintervals occurring in the Cantor construction belong to C.*)

12. Prove that every perfect set $X \subset \mathbb{R}$ is uncountable. (*Hint: If X is countable, construct a descending sequence of bounded, closed subsets of X whose intersection is empty.*)

13. A subset A of $\mathbb{R}$ is said to be nowhere dense in $\mathbb{R}$ provided that every open set U has an open subset that is disjoint from A. Show that the Cantor set is nowhere dense in $\mathbb{R}$.

14. Show that a strictly increasing function that is defined on an interval has a continuous inverse.

15. Let f be a continuous function and B be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (*Hint: The collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets.*)

16. Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.