Fall 2022 - Real Analysis
Homework 5

1. Suppose f and g are continuous functions on an open set $U \subset \mathbb{R}^n$. Show that if $f = g$ a.e. on U then, in fact, $f = g$ on U. Is a similar assertion true if the open set U is replaced by a general measurable set E?

2. Let D and E be measurable sets and f a function with domain $D \cup E$. We proved that f is measurable on $D \cup E$ if and only if its restrictions to D and E are measurable. Is the same true if "measurable" is replaced by "continuous"?

3. Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable?

4. (a) Suppose f is a real-valued function on $\mathbb{R}$ such that $\{f > r\}$ is measurable for each rational number r. Is f necessarily measurable?

(b) Suppose f is a real-valued function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each real number c. Is f necessarily measurable?

5. Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable. (*Hint: The collection of sets A that have the property that $f^{-1}(A)$ is measurable is σ-algebra.*)

6. (Borel measurability) A function f is said to be **Borel measurable** provided its domain E is a Borel set and for each $c \in \mathbb{R}$, the set $\{x \in E : f(x) > c\}$ is a Borel set. Verify that Proposition 1 and Theorem 6 remain valid if we replace "(Lebesgue) measurable set" by "'Borel set." Show that:

(1) Every Borel measurable function is Lebesgue measurable.
(2) If f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set.
(3) If f and g are Borel measurable, so is $f \circ g$.
(4) If f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

7. Suppose f and g are real-valued functions defined on all of $\mathbb{R}$, f is measurable, and g is continuous. Is the composition $f \circ g$ necessarily measurable?

8. Let f be a measurable function and g be a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable (*Hint.* You can start by proving that a Lipschitz function maps a set of measure 0 to a set of measure 0 and an F_σ set to an F_σ set and deduce that it maps a measurable set to a measurable set).

9. Show that if $E \subset \mathbb{R}$ is measurable and $f : E \longrightarrow \mathbb{R}$ is monotone increasing on E, then f is measurable.

10. Let f be a bounded measurable function on a set E. Show that there are sequences of simple functions on E, $\{\phi_n\}_n$ and $\{\psi_n\}_n$, such that $\{\phi_n\}_n$ is increasing and $\{\psi_n\}_n$ is decreasing and each of these sequences converges to f uniformly on E.

11. Let f be a measurable function on E that is finite a.e.on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \backslash F) < \epsilon$.

12. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. Show that for each $\epsilon > 0$, there is a measurable set F contained in E and a sequence $\{\phi_n\}_n$ of simple functions on E such that $\phi_n \longrightarrow f$ uniformly on F and $m(E \backslash F) < \epsilon$. (*Hint: See the preceding problem.*)

13. Show that the sum and product of two simple functions are simple as are the max and the min.

14. Let A and B be any sets. Show that

$$\chi_{A \cap B} = \chi_A \cdot \chi_B, \quad \chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B, \quad \chi_{A^c} = 1 - \chi_A$$

15. (**Dini's Theorem**) Let $\{f_n\}_n$ be an increasing sequence of continuous functions on the interval $[a,\ b]$ which converges pointwise on $[a,\ b]$ to the continuous function f on $[a,\ b]$. Show that the convergence is uniform on $[a,\ b]$. (*Hint: Let $\epsilon > 0$. For each natural number n, define $E_n = \{x \in [a,\ b] : \ |f(x) - f_n(x)| < \epsilon\}$. Show that $\{E_n\}_n$ is a (relatively) open cover of $[a,\ b]$ and use the Heine-Borel Theorem.*)

16. Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be increasing. Show that f is measurable by first showing that, for each natural number n, the strictly increasing function $f(x) = f(x) + \dfrac{x}{n}$ is measurable, and then taking pointwise limits