Fall 2022 - Real Analysis
Homework 6

1. Suppose f is a function that is continuous on a closed set $F \subset \mathbb{R}$. Show that f has a continuous extension to all of $\mathbb{R}$. This is a special case of the forthcoming Tietze Extension Theorem. (*Hint: Express* $\mathbb{R} \backslash F$ *as the union of a countable disjoint collection of open intervals and define* f *to be linear on the closure of each of these intervals.*)

2. Show that the conclusion of Egorov's Theorem can fail if we drop the assumption that the domain has finite measure.

3. Let $\{f_n\}_n$ be a sequence of measurable functions on a set E that converges to the real-valued function f pointwise on E. Show that $E = \bigcup_{k=1}^{\infty} E_k$, where for each index k, E_k is measurable, and $\{f_n\}_n$ converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.

4. A partition $\mathcal{P}'$ of $[a,\ b]$ is called a refinement of a partition $\mathcal{P}$ provided each partition point of $\mathcal{P}$ is also a partition point of $\mathcal{P}'$. For a bounded function f on $[a,\ b]$, show that under refinement lower Darboux sums increase and upper Darboux sums decrease.

5. Use the preceding problem to show that for a bounded function on a closed, bounded interval, each lower Darboux sum is no greater than each upper Darboux sum. From this conclude that the lower Riemann integral is no greater than the upper Riemann integral.

6. Suppose the bounded function f on $[a,\ b]$ is Riemann integrable over $[a,\ b]$. Show that there is a sequence $\{\mathcal{P}_n\}_n$ of partitions of $[a,\ b]$ for which $\lim_{n\to\infty}[U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)] = 0$.

7. Let f be a bounded function on $[a,\ b]$. Suppose there is a sequence $\{\mathcal{P}_n\}_n$ of partitions of $[a,\ b]$ for which $\lim_{n\to\infty}[U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n)] = 0$. Show that f is Riemann integrable over $[a,\ b]$.

8. Use the preceding problem to show that since a continuous function f on a closed, bounded interval $[a,\ b]$ is uniformly continuous on $[a,\ b]$, it is Riemann integrable over $[a,\ b]$.

9. Let f be an increasing real-valued function on $[0,\ 1]$. For a natural number n, define $\mathcal{P}_n$ to be the partition of $[0,\ 1]$ into n subintervals of length $1/n$. Show that $U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) \leq \frac{f(1) - f(0)}{n}$. Use Problem 7 to show that f is Riemann integrable over $[0,\ 1]$.

10. Let $\{f_n\}_n$ be a sequence of bounded functions that converges uniformly to f on the closed, bounded interval $[a,\ b]$. If each f_n is Riemann integrable over $[a,\ b]$, show that f also is Riemann integrable over $[a,\ b]$. Is it true that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$?