Fall 2022 - Real Analysis
Homework 7

1. Let $E \subset \mathbb{R}^n$ have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f dx = 0$.

2. Let f be a bounded measurable function on a set of finite measure $E \subset \mathbb{R}^n$. For a measurable subset A of E, show that $\int_A f dx = \int_E f \chi_A dx$.

3. Does the Bounded Convergence Theorem (Theorem 7 of Lecture 13) hold for the Riemann integral?

4. Let f be a bounded measurable function on a set E of finite measure. Assume g is bounded and $f = g$ a.e. on E. Show that $\int_E f dx = \int_E g dx$.

5. Does the Bounded Convergence Theorem hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{f_n\}_n$ is uniformly bounded on E?

6. Let f be a nonnegative bounded measurable function on a set of finite measure E. Assume $\int_E f dx = 0$. Show that $f = 0$ a.e. on E.

7. Let E be a set of measure zero and define $f = \infty$ on E. Show that $\int_E f dx = 0$.

8. For a real number a, define $f(x) = x^a$ for $0 < x \leq 1$, and $f(0) = 0$. Compute $\int_E f dx$ where $E = [0,\ 1]$. Note that this problem is easy to solve if we use the Fundamental Theorem of Calculus. Since we haven't yet covered the FTC, we can solve the problem by using the definition of the integral.

For $n \in \mathbb{N}$, consider the subdivision of the interval $[(1/2^n),\ 1]$ by the points

$$x_{n,k} = \frac{1}{2^{\frac{n2^n - k}{2^n}}}, \qquad k = 0, 1, \cdots, n2^n.$$

Let $\phi_n : [0,\ 1] \longrightarrow \mathbb{R}$ be given by $\phi_n = \sum_{k=1}^{n2^n} x_{n,k}^a \, \chi_{(x_{n,k-1},\ x_{n,k}]}$.

(1) Show that if $a \geq 0$, then the sequence $\{\phi_n\}$ is uniformly bounded and $\phi_n \longrightarrow f$ pointwise.
(2) Show that if $a < 0$, then the sequence $\{\phi_n\}$ is increasing and $\phi_n \longrightarrow f$ pointwise.
(3) Show that $\lim_{n\to\infty} \int_{[0,1]} \phi_n dx = \int_{[0,1]} f dx$
(4) Use a formula for a geometric sum to compute $\int_{[0,1]} \phi_n dx$.
(5) Find $\int_{[0,1]} f dx$ (consider separate cases $a > -1$, $a < -1$, and $a = -1$).

Hint: You might need to use the following

$$\lim_{y\to 1} \frac{1-y}{1-y^{a+1}} = \frac{1}{a+1} \quad \text{(L'Hopital's Rule)}.$$

9. Let $\{f_n\}_n$ be a sequence of nonnegative measurable functions that converges to f pointwise on a set E. Let $M > 0$ be such that $\int_E f_n dx \leq M$ for all n. Show that $\int_E f dx \leq M$.

10. Let the function f be nonnegative and integrable over E and $\epsilon > 0$. Show there is a simple function ϕ on E that has finite support, $0 \leq \phi \leq f$ on E and $\int_E (f - \phi) dx < \epsilon$. *Hint:* Luzin's Theorem might be useful.

11. Let $\{f_n\}_n$ be a sequence of nonnegative measurable functions on $\mathbb{R}^q$ that converges pointwise on $\mathbb{R}^q$ to f and f be integrable over $\mathbb{R}^q$. Show that if $\int_{\mathbb{R}^q} f dx = \lim_{n\to\infty} \int_{\mathbb{R}^q} f_n dx$, then $\int_E f dx = \lim_{n\to\infty} \int_E f_n dx$ for any measurable set $E \subset \mathbb{R}^q$.

12. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \le x < n+1$. Show that $\int_E f dx = \sum_{n=1}^{\infty} a_n$.

13. Let f be a nonnegative measurable function on E.

- Show there is an increasing sequence $\{\phi_n\}$ of nonnegative simple functions on E, each of finite support, which converges pointwise on E to f.
- Show that $\int_E f dx = \sup\left\{\int_E \phi dx : \ \phi \text{ simple, of finite support and } 0 \le \phi \le f \text{ on } E\right\}$.

14. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \le f$ on E for each n. Show that $\lim_{n\to\infty} \int_E f_n dx = \int_E f dx$.