Fall 2022 - Real Analysis
Homework 8

1. For a measurable function f on $[1,\ \infty)$ which is bounded on bounded sets, define $a_n = \displaystyle\int_n^{n+1} f(x)dx$ for each natural number n. Is it true that f is integrable over $[1,\ \infty)$ if and only if the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1,\ \infty)$ if and only if the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely?

2. Let g be a nonnegative integrable function over $E \subset \mathbb{R}^q$ and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n, $|f_n| \leq g$ a.e.on E. Show that

$$\int_E \liminf_{n\to\infty} f_n dx \leq \liminf_{n\to\infty} \int_E f_n dx \leq \limsup_{n\to\infty} \int_E f_n dx \leq \int_E \limsup_{n\to\infty} f_n dx.$$

Hint: You can use Lebesgue Dominated Convergence Theorem and also if $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $\{b_n\}$ converges to b and $b_n \leq a_n$ for all n, then $b \leq \liminf a_n$.

3. Let f be a measurable function on $E \subset \mathbb{R}^q$ which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative on E. Define $\displaystyle\int_E f dx = \int_E g dx + \int_E h dx$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function g and nonnegative function h whose sum is f.
Hint: Think about decomposing a function F as $F = F^+ - F^-$.

4. Let $\{f_n\}$ be a sequence of integrable functions on $E \subset \mathbb{R}^q$ for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\displaystyle\int_E |f_n - f|\, dx \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| dx = \int_E |f| dx$.
Hint: Use the **General Lebesgue Dominated Convergence Theorem***: Let $\{f_n\}_n$ be a sequence of measurable functions on a set E such that $f_n \longrightarrow f$ a.e. on E. Suppose that there exists a sequence of functions $\{g_n\}_n$ on E such that $g_n \geq 0$ and $g_n \longrightarrow g$ a.e. on E and $|f_n| \leq g_n$ for all n. If $\displaystyle\lim_{n\to\infty} \int_E g_n dx = \int_E g dx$, then $\displaystyle\lim_{n\to\infty} \int_E f_n dx = \int_E f dx$.*

5. Let f be a nonnegative measurable function on $\mathbb{R}$. Show that $\displaystyle\lim_{n\to\infty} \int_{-n}^{n} f dx = \int_{\mathbb{R}} f dx$.

6. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) :\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y. Suppose for each fixed value of x, $\displaystyle\lim_{y\to 0} f(x, y) = f(x)$ and that for all y, we have $|f(x, y)| \leq g(x)$ where g is integrable over $[0,\ 1]$. Show that

$$\lim_{y\to 0} \int_0^1 f(x, y)dx = \int_0^1 f(x)dx\,.$$

Also show that if the function $f(x, y)$ is continuous in y for each x, then $h(y) = \displaystyle\int_0^1 f(x, y)dx$ is a continuous function of y.

7. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) :\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y. For each $(x, y) \in Q$ let the partial derivative a $\dfrac{\partial f}{\partial y}$ exist. Suppose that

- There is $a \in [0,\ 1]$ such that $\alpha(x) = f(x, a)$ is integrable over $[0,\ 1]$
- There is a function g that is integrable over $[0,\ 1]$ and such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq g(x) \quad \text{for all} \quad (x, y) \in Q\,.$$

- The function f satisfies the Fundamental Teorem of Calculus with respect to y. That is

$$f(x, y) - f(x, p) = \int_p^y \frac{\partial f}{\partial y}(x, t)dt \quad \forall p \in [0,\ 1]$$

Prove the followings:

- For every $y \in [0,\ 1]$, the function $f(x, y)$ (as a function of x) is integrable over $[0,\ 1]$
- For all $y \in [0,\ 1]$

$$\frac{d}{dy}\left[\int_0^1 f(x,y)dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x,y)dx$$

8. Let f be a integrable function on $E \subset \mathbb{R}^q$. Show that for each $\epsilon > 0$, there is a natural number N for which if $n \geq N$, then $\displaystyle\int_{E_n} f dx < \epsilon$ where $E_n = \{x \in E :\ |x| > n\}$.

9. For each of the two functions f on $[1,\ \infty)$ defined below, show that $\lim_{n\to\infty} \int_1^n f(x)dx$ exists while f is not integrable over $[0,\ \infty)$. Does this contradict the continuity of integration?

- Define $f(x) = (-1)^n/n$, for $n \leq x < n+1$.
- Define $f(x) = (\sin x)/x$ for $1 < x < \infty$. *Hint:* you can first show that

$$\int_{2j\pi}^{2j\pi+2\pi} \frac{\sin x}{x} dx = \int_0^\pi \frac{\sin x}{x + 2j\pi} dx - \int_0^\pi \frac{\sin x}{x + (2j+1)\pi} dx$$