Fall 2022 - Real Analysis
Homework 9

1. Consider the two iterated integrals

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} \frac{x}{1-y^2} dxdy \quad\text{and}\quad I_2 = \int_{-1}^{1}\int_{-1}^{1} \frac{x}{1-y^2} dydx$$

Show that I_1 exists but I_2 does not (Note that the function $x/(1-y^2)$ is unbounded in the square $(-1,\ 1)^2$.

2. Show that

$$\int_1^\infty \left[\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2} dy\right] dx = \frac{-\pi}{4}; \quad \int_1^\infty \left[\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2} dx\right] dy = \frac{\pi}{4}; \quad \int_1^\infty \int_1^\infty \frac{\left|x^2-y^2\right|}{(x^2+y^2)^2} dxdy = \infty.$$

Hint: The function $\dfrac{x^2-y^2}{(x^2+y^2)^2}$ is continuous and bounded on $[1,\ \infty)\times[1,\ \infty)$ and $\dfrac{d}{dy}\left[\dfrac{y}{x^2+y^2}\right] = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ and $\dfrac{d}{dx}\left[\dfrac{x}{x^2+y^2}\right] = -\dfrac{x^2-y^2}{(x^2+y^2)^2}$. For the third integral you can use polar coordinates.

3. Given $f\in\mathcal{L}(\mathbb{R})$, define the function g on $\mathbb{R}$ by $g(x) = \int_{-\infty}^{x} f(t)dt$. Let $a\in\mathbb{R}$ be fixed. Show that the function h defined by $h(x) = g(x+a) - g(x)$ is an integrable function of x and

$$\int_{-\infty}^{\infty} h(x)dx = a\int_{-\infty}^{\infty} f(x)dx$$

4. Gaussian integral. The aim of this problem is to show that $I = \int_0^\infty \mathrm{e}^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

- Method 1. (Seen in Multivariable Calculus) Verify that $\mathrm{e}^{-x^2}\in\mathcal{L}(0,\ \infty)$ and use Fubini's Theorem and polar coordinates to prove that

$$I^2 = \int_0^\infty\int_0^\infty \mathrm{e}^{-(x^2+y^2)} dxdy = \int_0^{\pi/2}\int_0^\infty r\mathrm{e}^{-r^2} dr\, d\theta = \frac{\pi}{4}.$$

- Method 2. (First proof given by Laplace) Use the change of variable $x = yt$ in the inner integral of $I^2 = \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-(x^2+y^2)} dx\right) dy$ to verify that

$$I^2 = \int_0^\infty \left(\int_0^\infty y\mathrm{e}^{-y^2(1+t^2)} dt\right) dy = \int_0^\infty \left(\int_0^\infty y\mathrm{e}^{-y^2(1+t^2)} dy\right) dt = \frac{\pi}{4}.$$

  Hint: $(\arctan x)' = \dfrac{1}{1+x^2}$

5. The aim of this problem is to show that the improper integral $\int_0^\infty \dfrac{\sin x}{x} dx = \dfrac{\pi}{2}$ and $\dfrac{\sin x}{x}\notin\mathcal{L}(0,\ \infty)$.

(1) Verify that for $x>0$, we have $\dfrac{1}{x} = \int_0^\infty \mathrm{e}^{-tx} dt$.

(2) Use the previous part to verify that for $C>0$

$$\int_0^C \frac{\sin x}{x} dx = \int_0^C \left[\int_0^\infty \frac{\mathrm{e}^{(i-t)x} - \mathrm{e}^{(-i-t)x}}{2i} dt\right] dx$$

Hint: use the fact that $\sin x = \dfrac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{2i}$.

(3) Use Fubini's Theorem to verify that for $M>0$

$$\int_0^C \left[\int_0^M \frac{\mathrm{e}^{(i-t)x} - \mathrm{e}^{(-i-t)x}}{2i} dt\right] dx = \frac{1}{2i}\int_0^M \left[\frac{\mathrm{e}^{(i-t)C}}{i-t} + \frac{\mathrm{e}^{(-i-t)C}}{i+t}\right] dt + \int_0^M \frac{dt}{1+t^2}$$

(4) Use the Dominated Convergence Theorem to show that

$$\lim_{C\to\infty}\left(\frac{1}{2i}\int_0^M \left[\frac{\mathrm{e}^{(i-t)C}}{i-t} + \frac{\mathrm{e}^{(-i-t)C}}{i+t}\right] dt + \int_0^M \frac{dt}{1+t^2}\right) = \int_0^M \frac{dt}{1+t^2}$$

(5) Deduce that $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ (as an improper integral).

(6) Show that $\dfrac{\sin x}{x} \notin \mathcal{L}(0,\ \infty)$. *Hint:* You can start by verifying that $\displaystyle\int_{j\pi}^{(j+1)\pi} \frac{|\sin x|}{x}\, dx = \int_0^{\pi} \frac{\sin x}{x + j\pi}\, dx$.

6. Let $E \subset \mathbb{R}^p$ and $f : E \longrightarrow \overline{\mathbb{R}}$ be a measurable function and f is finite a.e. on E. The *distribution function of f over E* is the function $\omega : [0,\ \infty) \longrightarrow [0,\ \infty]$ given by $\omega(t) = m\left(\{|f| > t\}\right)$ (this notion is related to probability). Show that:

(1) The function ω is monotone deceasing over $[0,\ \infty)$.
(2) $\lim\limits_{t\to\infty} \omega(t) = 0$.
(3) The function ω is continuous from the right: For every $a \geq 0$, $\lim\limits_{t\to a^+} \omega(t) = \omega(a)$.
(4) $\lim\limits_{t\to a^-} \omega(t) = m\left(\{|f| \geq a\}\right)$.
(5) For a.e. $x \in E$, we have $\displaystyle\int_0^{\infty} \chi_{\{|f|>t\}}(x) dt = |f(x)|$.
(6) If $f \in \mathcal{L}(E)$, we have $\displaystyle\int_0^{\infty} \omega(t) dt = \int_E |f| dx$ (*Hint*: You can use Fubini's Theorem).
(7) $f \in \mathcal{L}(E) \Longleftrightarrow \omega \in \mathcal{L}(0,\ \infty)$ (*Hint*: Consider the function g defined on $E \times \mathbb{R}^+$ by $g(x,t) = \chi_{\{|f|>t\}}(x)$ and use Tonelli's Theorem.

7. Let $f(x) = \mathrm{e}^{-|x|}$, $g(x) = \mathrm{e}^{-x^2}$, and $h(x) = x\mathrm{e}^{-x^2}$. Compute $f * f$, $g * g$ and $h * h$.

8. Prove the following properties of the convolution $*$. For $f, h, g \in \mathcal{L}(\mathbb{R}^n)$:

(1) $f * g = g * f$;
(2) $(f * g) * h = f * (g * h)$;
(3) $f * T_a(g) = T_a(f) * g = T_a(f * g)$, where T_a is the translation operator defined on a function f by $T_a(f)(x) = f(x + a)$.

9. Let $f, g \in \mathcal{L}(\mathbb{R})$ with g bounded. Prove the following

(1) $f * g$ is a continuous function on $\mathbb{R}$;
(2) $f * g$ is a bounded function on $\mathbb{R}$ and $\|f * g\|_{\infty} \leq \|f\|_1\ \|g\|_{\infty}$ i.e.

$$\sup\{|f * g(x)| :\ x \in \mathbb{R}\} \leq \int_{\mathbb{R}} |f| dx\ \sup\{|g(x)| :\ x \in \mathbb{R}\}$$

Hint: The following result could be useful. Let $f \in \mathcal{L}(\mathbb{R})$, then for any $\epsilon > 0$, there exists $A > 0$ and continuous function h in $\mathbb{R}$ with $h(x) = 0$ for $|x| > A$ and such that $\displaystyle\int_{-\infty}^{\infty} |f - h|\, dx < \epsilon$.