I. Hölder continuous solutions of a class of planar vector fields

Hamid Meziani

Florida International University

Geometric Analysis of PDE and Several Complex Variables, Serra Negra, SP, Brazil, 2015

References about solvability of systems of vector fields

- S. Berhanu, P. Cordaro, J. Hounie: *An introduction to involutive structures*, New Math. Mono., 6, Cambridge University Press, Cambridge, (2008)
- F. Treves, *Hypo-Analytic Structures: Local Theory*, Princeton Mathematical Series, 40, Princeton Univ. Press, NJ (1992)

References about solvability of systems of vector fields

- S. Berhanu, P. Cordaro, J. Hounie: *An introduction to involutive structures*, New Math. Mono., 6, Cambridge University Press, Cambridge, (2008)
- F. Treves, *Hypo-Analytic Structures: Local Theory*, Princeton Mathematical Series, 40, Princeton Univ. Press, NJ (1992)

References about solvability of systems of vector fields

- S. Berhanu, P. Cordaro, J. Hounie: *An introduction to involutive structures*, New Math. Mono., 6, Cambridge University Press, Cambridge, (2008)
- F. Treves, *Hypo-Analytic Structures: Local Theory*, Princeton Mathematical Series, 40, Princeton Univ. Press, NJ (1992)

Discuss a class of planar vector fields in $\mathbb{R}^2$ that have analogous properties as those of $\overline{\partial}$.

- Hölder solvability:
  $\overline{\partial} u = f,\ f \in L^p,\ p > 2 \ \Rightarrow\ u \in C^\alpha,\ \alpha = \dfrac{p-2}{p}$
- Integral representation: $u = \dfrac{-1}{\pi} \displaystyle\int_\Omega \dfrac{f(\zeta}{\zeta - z}\, d\xi d\eta + H(z)$
- Similarity Principle: $\overline{\partial} u = Au + B\overline{u} \ \Rightarrow\ u(z) = H(z)\mathrm{e}^{s(z)}$
- Boundary value problem (RH):
  $\overline{\partial} u = Au + B\overline{u} + f$ in Ω, $\mathrm{Re}(\lambda u) = \phi$ on $\partial\Omega$.

Discuss a class of planar vector fields in $\mathbb{R}^2$ that have analogous properties as those of $\overline{\partial}$.

- Hölder solvability:
  $$\overline{\partial} u = f,\ f \in L^p,\ p > 2 \ \Rightarrow\ u \in C^\alpha,\ \alpha = \frac{p-2}{p}$$
- Integral representation: $u = \frac{-1}{\pi} \int_\Omega \frac{f(\zeta}{\zeta - z} d\xi d\eta + H(z)$
- Similarity Principle: $\overline{\partial} u = Au + B\overline{u} \ \Rightarrow\ u(z) = H(z)\mathrm{e}^{s(z)}$
- Boundary value problem (RH):
  $\overline{\partial} u = Au + B\overline{u} + f$ in Ω, $\mathrm{Re}(\lambda u) = \phi$ on $\partial\Omega$.

Discuss a class of planar vector fields in $\mathbb{R}^2$ that have analogous properties as those of $\overline{\partial}$.

- Hölder solvability:
  $\overline{\partial} u = f,\ f \in L^p,\ p > 2 \ \Rightarrow\ u \in C^\alpha,\ \alpha = \dfrac{p-2}{p}$
- Integral representation: $u = \dfrac{-1}{\pi} \int_\Omega \dfrac{f(\zeta}{\zeta - z} d\xi d\eta + H(z)$
- Similarity Principle: $\overline{\partial} u = Au + B\overline{u} \ \Rightarrow\ u(z) = H(z)\mathrm{e}^{s(z)}$
- Boundary value problem (RH):
  $\overline{\partial} u = Au + B\overline{u} + f$ in Ω, $\mathrm{Re}(\lambda u) = \phi$ on $\partial\Omega$.

Discuss a class of planar vector fields in $\mathbb{R}^2$ that have analogous properties as those of $\overline{\partial}$.

- Hölder solvability:
  $$\overline{\partial}u = f,\ f \in L^p,\ p > 2 \ \Rightarrow\ u \in C^\alpha,\ \alpha = \frac{p-2}{p}$$
- Integral representation: $u = \dfrac{-1}{\pi}\displaystyle\int_\Omega \frac{f(\zeta}{\zeta - z}\, d\xi d\eta + H(z)$
- Similarity Principle: $\overline{\partial}u = Au + B\overline{u} \ \Rightarrow\ u(z) = H(z)\mathrm{e}^{s(z)}$
- Boundary value problem (RH):
  $\overline{\partial}u = Au + B\overline{u} + f$ in Ω, $\mathrm{Re}(\lambda u) = \phi$ on $\partial\Omega$.

Discuss a class of planar vector fields in $\mathbb{R}^2$ that have analogous properties as those of $\overline{\partial}$.

- Hölder solvability:
  $\overline{\partial} u = f,\ f \in L^p,\ p > 2 \ \Rightarrow\ u \in C^\alpha,\ \alpha = \dfrac{p-2}{p}$
- Integral representation: $u = \dfrac{-1}{\pi} \displaystyle\int_\Omega \frac{f(\zeta}{\zeta - z} \, d\xi d\eta + H(z)$
- Similarity Principle: $\overline{\partial} u = Au + B\overline{u} \ \Rightarrow\ u(z) = H(z)\mathrm{e}^{s(z)}$
- Boundary value problem (RH):
  $\overline{\partial} u = Au + B\overline{u} + f$ in $\Omega\,,$ $\mathrm{Re}(\lambda u) = \phi$ on $\partial\Omega$.

Necessary and Sufficient condition for local solvability given by Nirenberg-Treves Condition ($\mathcal{P}$)

For nonsingular planar vector fields L. We can find local coordinates such that

$$L = m(x,t)\left(\frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}\right)$$

with b an $\mathbb{R}$-valued function. Condition ($\mathcal{P}$) just means that for every x, the function $t \longrightarrow b(x,t)$ does not change sign.

Example. $\dfrac{\partial}{\partial t} - ix^n t^m \dfrac{\partial}{\partial x}$ with $m \in \mathbb{Z}^+$ satisfies ($\mathcal{P}$) iff m is even.

Necessary and Sufficient condition for local solvability given by Nirenberg-Treves Condition $(\mathcal{P})$
For nonsingular planar vector fields L. We can find local coordinates such that

$$L = m(x,t)\left(\frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}\right)$$

with b an $\mathbb{R}$-valued function. Condition $(\mathcal{P})$ just means that for every x, the function $t \longrightarrow b(x,t)$ does not change sign.

Example. $\dfrac{\partial}{\partial t} - ix^n t^m \dfrac{\partial}{\partial x}$ with $m \in \mathbb{Z}^+$ satisfies $(\mathcal{P})$ iff m is even.

Necessary and Sufficient condition for local solvability given by Nirenberg-Treves Condition ($\mathcal{P}$)
For nonsingular planar vector fields L. We can find local coordinates such that

$$L = m(x,t)\left(\frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}\right)$$

with b an $\mathbb{R}$-valued function. Condition ($\mathcal{P}$) just means that for every x, the function $t \longrightarrow b(x,t)$ does not change sign.

Example. $\frac{\partial}{\partial t} - ix^n t^m \frac{\partial}{\partial x}$ with $m \in \mathbb{Z}^+$ satisfies ($\mathcal{P}$) iff m is even.

Necessary and Sufficient condition for local solvability given by Nirenberg-Treves Condition ($\mathcal{P}$)
For nonsingular planar vector fields L. We can find local coordinates such that

$$L = m(x,t)\left(\frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}\right)$$

with b an $\mathbb{R}$-valued function. Condition ($\mathcal{P}$) just means that for every x, the function $t \longrightarrow b(x,t)$ does not change sign.

Example. $\dfrac{\partial}{\partial t} - ix^n t^m \dfrac{\partial}{\partial x}$ with $m \in \mathbb{Z}^+$ satisfies ($\mathcal{P}$) iff m is even.

If L satisfies $(\mathcal{P})$, then it is locally integrable.

Given point p, there is U open $p \in U$ and $Z : U \longrightarrow \mathbb{C}$ such that

$$LZ = 0 \quad \text{and} \quad dZ \neq 0 \ \text{ in } \ U$$

In fact we can assume

$$Z(x,t) = x + i\phi(x,t)$$

$$L = m\left((1 + i\phi_x)\frac{\partial}{\partial x} - i\phi_t \frac{\partial}{\partial t}\right)$$

If L satisfies $(\mathcal{P})$, then it is locally integrable.
Given point p, there is U open $p \in U$ and $Z : U \longrightarrow \mathbb{C}$ such that

$$LZ = 0 \quad \text{and} \quad dZ \neq 0 \;\text{ in }\; U$$

In fact we can assume

$$Z(x,t) = x + i\phi(x,t)$$

$$L = m\left((1 + i\phi_x)\frac{\partial}{\partial x} - i\phi_t\frac{\partial}{\partial t}\right)$$

If L satisfies $(\mathcal{P})$, then it is locally integrable.
Given point p, there is U open $p \in U$ and $Z : U \longrightarrow \mathbb{C}$ such that

$$LZ = 0 \quad \text{and} \quad dZ \neq 0 \ \text{ in } \ U$$

In fact we can assume

$$Z(x,t) = x + i\phi(x,t)$$

$$L = m\left((1 + i\phi_x)\frac{\partial}{\partial x} - i\phi_t \frac{\partial}{\partial t}\right)$$

If L satisfies $(\mathcal{P})$, then it is locally integrable.
Given point p, there is U open $p \in U$ and $Z : U \longrightarrow \mathbb{C}$ such that

$$LZ = 0 \quad \text{and} \quad dZ \neq 0 \ \text{ in } \ U$$

In fact we can assume

$$Z(x,t) = x + i\phi(x,t)$$

$$L = m\left((1 + i\phi_x)\frac{\partial}{\partial x} - i\phi_t\frac{\partial}{\partial t}\right)$$

If L satisfies $(\mathcal{P})$, then it is locally integrable.
Given point p, there is U open $p \in U$ and $Z : U \longrightarrow \mathbb{C}$ such that

$$LZ = 0 \quad \text{and} \quad dZ \neq 0 \ \text{ in } \ U$$

In fact we can assume

$$Z(x,t) = x + i\phi(x,t)$$

$$L = m\left((1 + i\phi_x)\frac{\partial}{\partial x} - i\phi_t\frac{\partial}{\partial t}\right)$$

We know if L satisfies ($\mathcal{P}$) and has C^∞ coefficients, then L is solvable in C^∞ category:

$$Lu = f \text{ has } C^\infty \text{ solutions if } f \in C^\infty .$$

An old result of Treves gives L^2-solvability. There is L^p-solvability in 2 variables under only Lipschitz regularity of coefficients of L [Hounie-Morales Melo] (97)

$$Lu = f \text{ has } L^p \text{ solutions if } f \in L^p .$$

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ satisfies ($\mathcal{P}$) with first integral $Z = x\mathrm{e}^{it}$. However $Lu = 0$ has singular solutions. For instance $u = \ln|x| + it$.

We know if L satisfies $(\mathcal{P})$ and has C^∞ coefficients, then L is solvable in C^∞ category:

$$Lu = f \text{ has } C^\infty \text{ solutions if } f \in C^\infty\,.$$

An old result of Treves gives L^2-solvability. There is L^p-solvability in 2 variables under only Lipschitz regularity of coefficients of L [Hounie-Morales Melo] (97)

$$Lu = f \text{ has } L^p \text{ solutions if } f \in L^p\,.$$

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ satisfies $(\mathcal{P})$ with first integral $Z = x\mathrm{e}^{it}$. However $Lu = 0$ has singular solutions. For instance $u = \ln|x| + it$.

We know if L satisfies $(\mathcal{P})$ and has C^∞ coefficients, then L is solvable in C^∞ category:

$$Lu = f \text{ has } C^\infty \text{ solutions if } f \in C^\infty .$$

An old result of Treves gives L^2-solvability. There is L^p-solvability in 2 variables under only Lipschitz regularity of coefficients of L [Hounie-Morales Melo] (97)

$$Lu = f \text{ has } L^p \text{ solutions if } f \in L^p .$$

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ satisfies $(\mathcal{P})$ with first integral $Z = xe^{it}$. However $Lu = 0$ has singular solutions. For instance $u = \ln|x| + it$.

We know if L satisfies $(\mathcal{P})$ and has C^∞ coefficients, then L is solvable in C^∞ category:

$$Lu = f \text{ has } C^\infty \text{ solutions if } f \in C^\infty .$$

An old result of Treves gives L^2-solvability. There is L^p-solvability in 2 variables under only Lipschitz regularity of coefficients of L [Hounie-Morales Melo] (97)

$$Lu = f \text{ has } L^p \text{ solutions if } f \in L^p .$$

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ satisfies $(\mathcal{P})$ with first integral $Z = x\mathrm{e}^{it}$. However $Lu = 0$ has singular solutions. For instance $u = \ln|x| + it$.

We know if L satisfies $(\mathcal{P})$ and has C^∞ coefficients, then L is solvable in C^∞ category:

$$Lu = f \text{ has } C^\infty \text{ solutions if } f \in C^\infty .$$

An old result of Treves gives L^2-solvability. There is L^p-solvability in 2 variables under only Lipschitz regularity of coefficients of L [Hounie-Morales Melo] (97)

$$Lu = f \text{ has } L^p \text{ solutions if } f \in L^p .$$

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ satisfies $(\mathcal{P})$ with first integral $Z = x\mathrm{e}^{it}$. However $Lu = 0$ has singular solutions. For instance $u = \ln|x| + it$.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies $(\mathcal{P})$ and the local first integrals are homeomorphisms.

Example. $\frac{\partial}{\partial t} - ix\frac{\partial}{\partial x}$ is not hypocomplex.

$\frac{\partial}{\partial t} - 3it^2\frac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies $(\mathcal{P})$ and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}\,, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}\,.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\,\}\,.$$

For f = characteristic function of S, the equation $Lu = f$ has no bounded solutions.

In order to gain regularity of solutions, it is necessary to assume that L does not have 1-dimensional orbits.

Assume that L is *hypocomplex*. L satisfies ($\mathcal{P}$) and the local first integrals are homeomorphisms.

Example. $\dfrac{\partial}{\partial t} - ix\dfrac{\partial}{\partial x}$ is not hypocomplex.

$\dfrac{\partial}{\partial t} - 3it^2\dfrac{\partial}{\partial x}$ is hypocomplex since its first integral $Z = x + it^3$ is global homeomorphism.

Hypocomplexity is not enough.

Example. [Berhanu-Cordaro-Hounie].

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}\,, \quad Z(x,t) = x + i\frac{|t|}{t}\exp\frac{-1}{|t|}\,.$$

L is a C^∞ hypocomplex vector field with first integral Z. Let

$$S = \{(x,t) : |Z(x,t)| < 1/2, \quad 0 < \arg(Z(x,t)) < \pi/4\}\,.$$

For $f =$ characteristic function of S, the equation $Lu = f$ has no bounded solutions.

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \ L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x,t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x,t)|^\sigma \leq C|\phi_t(x,t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\mathrm{dis}(Z(m), Z(\Sigma))^\sigma \leq C|\mathrm{Im}(Z_x\overline{Z}_y)|\,.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \; L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x,t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x,t)|^{\sigma} \leq C|\phi_t(x,t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\text{dis}(Z(m), Z(\Sigma))^{\sigma} \leq C|\text{Im}(Z_x \overline{Z}_y)|\,.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \; L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x,t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x,t)|^\sigma \leq C|\phi_t(x,t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\mathrm{dis}(Z(m), Z(\Sigma))^\sigma \leq C\left|\mathrm{Im}(Z_x \overline{Z}_y)\right|.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \ L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x,t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x,t)|^\sigma \leq C|\phi_t(x,t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\text{dis}(Z(m), Z(\Sigma))^\sigma \leq C\left|\text{Im}(Z_x\overline{Z}_y)\right|.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p: \ L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x, t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x, t)|^\sigma \leq C|\phi_t(x, t)|$$

for some $C > 0$ and $\sigma \in (0, \ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\mathrm{dis}(Z(m), Z(\Sigma))^\sigma \leq C|\mathrm{Im}(Z_x\overline{Z}_y)|\,.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \ L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x, t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x, t)|^\sigma \leq C|\phi_t(x, t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\mathrm{dis}(Z(m), Z(\Sigma))^\sigma \leq C|\mathrm{Im}(Z_x\overline{Z}_y)|\,.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \ L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x,t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x,t)|^\sigma \leq C|\phi_t(x,t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\mathrm{dis}(Z(m), Z(\Sigma))^\sigma \leq C|\mathrm{Im}(Z_x\overline{Z}_y)|\,.$$

Łojasiewicz type condition

Given L, let Σ be the characteristic set of L

$$\Sigma = \{p : \ L_p \wedge \overline{L}_p = 0\}$$

L satisfies the Łojasiewicz type condition (LC) if for every $p \in \Sigma$, L has a local first integral $Z = x + i\phi(x,t)$ (with $Z(p) = 0$) defined in an open set $p \in U \subset \mathbb{R}^2$such that Z is a homeomorphism and satisfyies

$$|\phi(x,t)|^{\sigma} \leq C|\phi_t(x,t)|$$

for some $C > 0$ and $\sigma \in (0,\ 1)$.

Remark. The (LC) condition is invariant under change of coordinates or choice of a local first integral.

Remark. If Σ is 1-dimensional manifold, then the (LC) condition can be written as

$$\mathrm{dis}(Z(m), Z(\Sigma))^{\sigma} \leq C\left|\mathrm{Im}(Z_x\overline{Z}_y)\right|.$$

Example. If L is of uniform finite type n along Σ, then L satisfies (LC) with $\sigma = \dfrac{n}{n+1}$.

Example. The vector field

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}$$

does not satisfy (LC) since

$$\lim_{t\to 0} t^2 \exp\frac{1-\sigma}{|t|} = \infty \quad \text{if } \sigma < 1\,.$$

Example. If L is of uniform finite type n along Σ, then L satisfies (LC) with $\sigma = \dfrac{n}{n+1}$.

Example. The vector field

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}$$

does not satisfy (LC) since

$$\lim_{t\to 0} t^2 \exp\frac{1-\sigma}{|t|} = \infty \quad \text{if } \sigma < 1\,.$$

Example. If L is of uniform finite type n along Σ, then L satisfies (LC) with $\sigma = \dfrac{n}{n+1}$.

Example. The vector field

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}$$

does not satisfy (LC) since

$$\lim_{t\to 0} t^2 \exp\frac{1-\sigma}{|t|} = \infty \quad \text{if } \sigma < 1\,.$$

Example. If L is of uniform finite type n along Σ, then L satisfies (LC) with $\sigma = \dfrac{n}{n+1}$.

Example. The vector field

$$L = \frac{\partial}{\partial t} - i\left(\frac{1}{t^2}\exp\frac{-1}{|t|}\right)\frac{\partial}{\partial x}$$

does not satisfy (LC) since

$$\lim_{t\to 0} t^2 \exp\frac{1-\sigma}{|t|} = \infty \quad \text{if } \sigma < 1\,.$$

Example. If $\phi(x,t)$ is homogeneous of degree $\lambda+1$ and $\phi_t(x,t) > 0$ for $(x,t) \neq (0,0)$, then the vector field associated with $Z = x + i\phi(x,t)$ satisfies (LC) with $\sigma = \lambda/\lambda+1$.
In polar coordinates

$$\phi(r,\theta) = r^{\lambda+1}Q(\theta) \quad \phi_t(r,\theta) = r^{\lambda}P(\theta)$$

with $P(\theta) > 0 \quad \forall\theta \in [0,\ 2\pi]$. Thus

$$|\phi(r,\theta)|^{\frac{\lambda}{\lambda+1}} \leq C|\phi_t(r,\theta)|$$

Example. If $\phi(x,t)$ is homogeneous of degree $\lambda+1$ and $\phi_t(x,t)>0$ for $(x,t)\neq(0,0)$,then the vector field associated with $Z=x+i\phi(x,t)$ satisfies (LC) with $\sigma=\lambda/\lambda+1$.
In polar coordinates

$$\phi(r,\theta)=r^{\lambda+1}Q(\theta)\quad \phi_t(r,\theta)=r^{\lambda}P(\theta)$$

with $P(\theta)>0\quad\forall\theta\in[0,\ 2\pi]$. Thus

$$|\phi(r,\theta)|^{\frac{\lambda}{\lambda+1}}\leq C|\phi_t(r,\theta)|$$

Example. If $\phi(x,t)$ is homogeneous of degree $\lambda+1$ and $\phi_t(x,t)>0$ for $(x,t)\neq(0,0)$,then the vector field associated with $Z=x+i\phi(x,t)$ satisfies (LC) with $\sigma=\lambda/\lambda+1$.
In polar coordinates

$$\phi(r,\theta)=r^{\lambda+1}Q(\theta)\quad \phi_t(r,\theta)=r^{\lambda}P(\theta)$$

with $P(\theta)>0\quad\forall\theta\in[0,\ 2\pi]$. Thus

$$|\phi(r,\theta)|^{\frac{\lambda}{\lambda+1}}\leq C|\phi_t(r,\theta)|$$

Example. If $\phi(x,t)$ is homogeneous of degree $\lambda+1$ and $\phi_t(x,t) > 0$ for $(x,t) \neq (0,0)$,then the vector field associated with $Z = x + i\phi(x,t)$ satisfies (LC) with $\sigma = \lambda/\lambda+1$.
In polar coordinates

$$\phi(r,\theta) = r^{\lambda+1}Q(\theta) \quad \phi_t(r,\theta) = r^{\lambda}P(\theta)$$

with $P(\theta) > 0 \quad \forall\theta \in [0,\ 2\pi]$. Thus

$$|\phi(r,\theta)|^{\frac{\lambda}{\lambda+1}} \leq C|\phi_t(r,\theta)|$$

Example. If $\phi(x,t)$ is homogeneous of degree $\lambda+1$ and $\phi_t(x,t)>0$ for $(x,t)\neq(0,0)$,then the vector field associated with $Z=x+i\phi(x,t)$ satisfies (LC) with $\sigma=\lambda/\lambda+1$.
In polar coordinates

$$\phi(r,\theta)=r^{\lambda+1}Q(\theta)\quad \phi_t(r,\theta)=r^{\lambda}P(\theta)$$

with $P(\theta)>0 \quad \forall\theta\in[0,\ 2\pi]$. Thus

$$|\phi(r,\theta)|^{\frac{\lambda}{\lambda+1}}\leq C|\phi_t(r,\theta)|$$

Example. If $\phi(x,t)$ is homogeneous of degree $\lambda+1$ and $\phi_t(x,t) > 0$ for $(x,t) \neq (0,0)$,then the vector field associated with $Z = x + i\phi(x,t)$ satisfies (LC) with $\sigma = \lambda/\lambda+1$.
In polar coordinates

$$\phi(r,\theta) = r^{\lambda+1}Q(\theta) \quad \phi_t(r,\theta) = r^{\lambda}P(\theta)$$

with $P(\theta) > 0 \quad \forall\theta \in [0,\ 2\pi]$. Thus

$$|\phi(r,\theta)|^{\frac{\lambda}{\lambda+1}} \leq C|\phi_t(r,\theta)|$$

Suppose that L is a vector field defined in an open set $\Omega' \subset \mathbb{R}^2$ s.t.

- L is hypocomplex;
- The characteristics Σ is 1-dimensional manifold;
- L satisfies (LC)

Suppose that L is a vector field defined in an open set $\Omega' \subset \mathbb{R}^2$ s.t.

- L is hypocomplex;
- The characteristics Σ is 1-dimensional manifold;
- L satisfies (LC)

Suppose that L is a vector field defined in an open set $\Omega' \subset \mathbb{R}^2$ s.t.

- L is hypocomplex;
- The characteristics Σ is 1-dimensional manifold;
- L satisfies (LC)

Suppose that L is a vector field defined in an open set $\Omega' \subset \mathbb{R}^2$ s.t.

- L is hypocomplex;
- The characteristics Σ is 1-dimensional manifold;
- L satisfies (LC)

Suppose that L is a vector field defined in an open set $\Omega' \subset \mathbb{R}^2$ s.t.

- L is hypocomplex;
- The characteristics Σ is 1-dimensional manifold;
- L satisfies (LC)

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'.
We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \quad \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'.
We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \quad \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'.
We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \quad \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'. We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \ \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'. We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \quad \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'.
We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \ \ \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'. We can assume that there exist

$$Z : \Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \ \ \forall p \in \Omega' .$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x} .$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Remark. L hypocomplex $\Rightarrow$ L has a global first integral in Ω'. We can assume that there exist

$$Z:\,\Omega' \longrightarrow Z(\Omega') \subset \mathbb{C}$$

a C^1 homeomorphism, such that

$$LZ = 0 \quad \text{and} \quad dZ(p) \neq \ \ \forall p \in \Omega'\,.$$

We can assume that

$$L = Z_x \frac{\partial}{\partial y} - Z_y \frac{\partial}{\partial x}\,.$$

Let $\Omega \subset\subset \Omega'$, $\Sigma = \sum_{j=1}^{N} \Sigma_j$. For each j let σ_j be the exponent of the (LC) on Σ_j. Let $\sigma = \max \sigma_j$.

Cauchy-Pompeiu operator

For $f \in L^1(\Omega)$ define

$$T_Z f(x,y) = \frac{1}{2\pi i} \int_\Omega \frac{f(\xi,\eta)\, d\xi d\eta}{Z(\xi,\eta) - Z(x,y)}\,.$$

Theorem ([C. Campana, P. Dattori, A.M.])

- *If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, then $\exists M = M(p, \sigma, \Omega)$ such that*

$$|T_Z f(x,y)| \leq M\,||f||_p\,, \quad \forall (x,y) \in \Omega\,.$$

- *If $f \in L^1(\Omega)$, then $T_Z f(x,y) \in L^q(\Omega)$ for any $1 < q < 2 - \dfrac{\sigma}{\sigma + 1}$.*

Theorem ([C. Campana, P. Dattori, A.M.])

- *If* $f \in L^p(\Omega)$ *with* $p > 2 + \sigma$, *then* $\exists M = M(p, \sigma, \Omega)$ *such that*

$$|T_Z f(x,y)| \leq M\,||f||_p\,, \quad \forall (x,y) \in \Omega\,.$$

- *If* $f \in L^1(\Omega)$, *then* $T_Z f(x,y) \in L^q(\Omega)$ *for any* $1 < q < 2 - \dfrac{\sigma}{\sigma+1}$.

Theorem ([C. Campana, P. Dattori, A.M.])

- *If* $f \in L^p(\Omega)$ *with* $p > 2 + \sigma$, *then* $\exists M = M(p, \sigma, \Omega)$ *such that*

$$|T_Z f(x,y)| \le M\,||f||_p\,, \quad \forall (x,y) \in \Omega\,.$$

- *If* $f \in L^1(\Omega)$, *then* $T_Z f(x,y) \in L^q(\Omega)$ *for any* $1 < q < 2 - \dfrac{\sigma}{\sigma+1}$.

Theorem ([C. Campana, P. Dattori, A.M.])

- *If* $f \in L^p(\Omega)$ *with* $p > 2 + \sigma$, *then* $\exists M = M(p, \sigma, \Omega)$ *such that*

$$|T_Z f(x,y)| \leq M \, ||f||_p \,, \quad \forall (x,y) \in \Omega \,.$$

- *If* $f \in L^1(\Omega)$, *then* $T_Z f(x,y) \in L^q(\Omega)$ *for any* $1 < q < 2 - \dfrac{\sigma}{\sigma + 1}$.

Theorem ([C. Campana, P. Dattori, A.M.])

- *If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, then $\exists M = M(p, \sigma, \Omega)$ such that*

$$|T_Z f(x, y)| \leq M\, ||f||_p\,, \quad \forall (x, y) \in \Omega\,.$$

- *If $f \in L^1(\Omega)$, then $T_Z f(x, y) \in L^q(\Omega)$ for any* $1 < q < 2 - \dfrac{\sigma}{\sigma + 1}$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2+\sigma$

$$|T_Z f(x,y)| \leq \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta)-Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2-\tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let
$s = \xi = \mathrm{Re} Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im} Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \leq (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2+\sigma$

$$|T_Z f(x,y)| \le \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2-\tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let

$s = \xi = \mathrm{Re}Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im}Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \le (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2+\sigma$

$$|T_Z f(x,y)| \leq \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta)-Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2-\tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let

$s = \xi = \mathrm{Re} Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im} Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \leq (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2 + \sigma$

$$|T_Z f(x,y)| \le \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2 - \tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let

$s = \xi = \mathrm{Re}Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im}Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \le (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2+\sigma$

$$|T_Z f(x,y)| \le \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta)-Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2-\tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let
$s = \xi = \mathrm{Re} Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im} Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \le (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2+\sigma$

$$|T_Z f(x,y)| \le \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta)-Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2-\tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let
$s = \xi = \mathrm{Re}Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im}Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \le (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Idea of the proof

Special case: $Z = x + i\dfrac{y|y|^\sigma}{\sigma+1}$ If $f \in L^p$ with $p > 2+\sigma$

$$|T_Z f(x,y)| \leq \frac{||f||_p}{\pi}\left(\int_\Omega \frac{d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|^q}\right)^{\frac{1}{q}} = \frac{||f||_p}{\pi} J^{\frac{1}{q}}$$

with $q = \dfrac{p}{p-1}$. Note that $q < 2-\tau$, $\tau = \dfrac{\sigma}{\sigma+1}$ let
$s = \xi = \mathrm{Re}Z$, $t = \dfrac{\eta|\eta|^\sigma}{1+\sigma} = \mathrm{Im}Z$.

$$J = (1+\sigma)^{\frac{1}{\sigma+1}} \int_{Z(\Omega)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q} \leq (1+\sigma)^{\frac{1}{\sigma+1}} \int_{D(z,R)} \frac{ds\, dt}{|t|^\tau\, |\zeta - z|^q}$$

with $\zeta = s + it$, $z = Z(x,y)$ and R is the diameter of $Z(\Omega)$.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \le M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \leq M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \leq M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \leq M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \le M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \le M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Lemma

$$\int_{D(z,R)} \frac{ds\,dt}{|t|^{\tau}\,|\zeta - z|^{q}} \leq M(q)\frac{R^{2-q-\tau}}{2-q-\tau}$$

This lemma together with the above estimate proves the special case. The general case can be brought into the special case by writing

$$J = \int_{\Omega'} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}} + \sum_{j}^{N} \int_{V_j} \frac{d\xi\,d\eta}{|Z(\xi,\eta) - Z(x,y)|^{q}}$$

with V_j tubular neighborhood of Σ_j and $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} V_j$. L is elliptic in Ω' and L can be transformed into the special case in each V_j with exponent σ_j.

Let $f \in L^1(\Omega)$. Let $g \in L^p(\Omega)$ with $p > 2 + \sigma$, then part (1) implies that

$$g_1(x,y) = \int_\Omega \frac{|g(\xi,\eta)|\, d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|} \in L^\infty(\Omega) \quad \text{and} \quad g_1 f \in L^1(\Omega)\,.$$

Then

$$\int_\Omega f g_1\, dxdy = \int_\Omega |g(\xi,\eta)| \int_\Omega \frac{|f(x,y)|\ dxdy}{|Z(x,y) - Z(\xi,\eta)|}\, d\xi d\eta$$

So

$$\int_\Omega \frac{|f(x,y)|\ dxdy}{|Z(x,y) - Z(\xi,\eta)|} \in L^q(\Omega) \quad \text{and} \quad T_Z f(x,y) \in L^q(\Omega)$$

Let $f \in L^1(\Omega)$. Let $g \in L^p(\Omega)$ with $p > 2 + \sigma$, then part (1) implies that

$$g_1(x,y) = \int_\Omega \frac{|g(\xi,\eta)|\, d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|} \in L^\infty(\Omega) \quad \text{and} \quad g_1 f \in L^1(\Omega)\,.$$

Then

$$\int_\Omega f g_1\, dxdy = \int_\Omega |g(\xi,\eta)| \int_\Omega \frac{|f(x,y)|\; dxdy}{|Z(x,y) - Z(\xi,\eta)|}\, d\xi d\eta$$

So

$$\int_\Omega \frac{|f(x,y)|\; dxdy}{|Z(x,y) - Z(\xi,\eta)|} \in L^q(\Omega) \quad \text{and} \quad T_Z f(x,y) \in L^q(\Omega)$$

Let $f \in L^1(\Omega)$. Let $g \in L^p(\Omega)$ with $p > 2 + \sigma$, then part (1) implies that

$$g_1(x,y) = \int_\Omega \frac{|g(\xi,\eta)|\, d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|} \in L^\infty(\Omega) \quad \text{and} \quad g_1 f \in L^1(\Omega)\,.$$

Then

$$\int_\Omega f g_1\, dxdy = \int_\Omega |g(\xi,\eta)| \int_\Omega \frac{|f(x,y)|\ dxdy}{|Z(x,y) - Z(\xi,\eta)|}\, d\xi d\eta$$

So

$$\int_\Omega \frac{|f(x,y)|\ dxdy}{|Z(x,y) - Z(\xi,\eta)|} \in L^q(\Omega) \quad \text{and} \quad T_Z f(x,y) \in L^q(\Omega)$$

Let $f \in L^1(\Omega)$. Let $g \in L^p(\Omega)$ with $p > 2 + \sigma$, then part (1) implies that

$$g_1(x,y) = \int_\Omega \frac{|g(\xi,\eta)|\, d\xi\, d\eta}{|Z(\xi,\eta) - Z(x,y)|} \in L^\infty(\Omega) \quad \text{and} \quad g_1 f \in L^1(\Omega)\,.$$

Then

$$\int_\Omega f g_1 \, dxdy = \int_\Omega |g(\xi,\eta)| \int_\Omega \frac{|f(x,y)|\ dxdy}{|Z(x,y) - Z(\xi,\eta)|}\, d\xi d\eta$$

So

$$\int_\Omega \frac{|f(x,y)|\ dxdy}{|Z(x,y) - Z(\xi,\eta)|} \in L^q(\Omega) \quad \text{and} \quad T_Z f(x,y) \in L^q(\Omega)$$

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^1(\Omega)$, then $LT_Z f(x,y) = f$

The proof is consequence of

Proposition

If $w \in C^0(\overline{\Omega}) \cap C^1(\Omega)$, then

$$w(x,y) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{w(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, dZ(\xi,\eta)$$

$$-\frac{1}{\pi}\int_{\Omega} \frac{Lw(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, d\xi d\eta\,.$$

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^1(\Omega)$, then $LT_Z f(x,y) = f$

The proof is consequence of

Proposition

If $w \in C^0(\overline{\Omega}) \cap C^1(\Omega)$, then

$$w(x,y) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{w(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, dZ(\xi,\eta)$$

$$-\frac{1}{\pi}\int_{\Omega} \frac{Lw(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, d\xi d\eta\,.$$

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^1(\Omega)$, *then* $LT_Z f(x,y) = f$

The proof is consequence of

Proposition

If $w \in C^0(\overline{\Omega}) \cap C^1(\Omega)$, *then*

$$w(x,y) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{w(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, dZ(\xi,\eta)$$

$$- \frac{1}{\pi} \int_{\Omega} \frac{Lw(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, d\xi d\eta\,.$$

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^1(\Omega)$, then $LT_Z f(x,y) = f$

The proof is consequence of

Proposition

If $w \in C^0(\overline{\Omega}) \cap C^1(\Omega)$, then

$$w(x,y) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{w(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, dZ(\xi,\eta)$$

$$- \frac{1}{\pi}\int_{\Omega} \frac{Lw(\xi,\eta)}{Z(\xi,\eta) - Z(x,y)}\, d\xi d\eta\,.$$

Proof of Theorem

Let $\phi \in \mathcal{D}(\Omega)$.

$$< LT_Z f(x,y), \phi > = - < T_Z f(x,y), L\phi >$$

$$= \int_\Omega f(\xi,\eta) \left(\frac{-1}{\pi} \int_\Omega \frac{L\phi(x,y)}{Z(x,y) - Z(\xi,\eta)} \, dxdy \right) d\xi d\eta$$

$$= < f, \phi > \ .$$

Proof of Theorem

Let $\phi \in \mathcal{D}(\Omega)$.

$$< LT_Z f(x,y), \phi > = - < T_Z f(x,y), L\phi >$$

$$= \int_\Omega f(\xi,\eta) \left(\frac{-1}{\pi} \int_\Omega \frac{L\phi(x,y)}{Z(x,y) - Z(\xi,\eta)} \, dxdy \right) d\xi d\eta$$

$$=< f, \phi > \; .$$

Remark. $f \in L^p$ with $p > 2 + \sigma$ is optimal for obtaining bounded solutions.

For the standard vector field $\dfrac{\partial}{\partial y} - i|y|^\sigma \dfrac{\partial}{\partial x}$ and

$$f(x,y) = \frac{-i|y|^\sigma}{\overline{Z(x,y)} \ln |Z(x,y)|} \in L^p(\mathbb{R}^2) \quad \text{for any } p < 2 + \sigma$$

but $f \notin L^{p'}$ with $p' > 2 + \sigma$. The function $u(x,y) = \ln |\ln Z(x,y)| \notin L^\infty$ and satisfies $Lu = f$. Equation $Lu = f$ has no bounded solutions.

Remark. $f \in L^p$ with $p > 2+\sigma$ is optimal for obtaining bounded solutions.
For the standard vector field $\dfrac{\partial}{\partial y} - i|y|^\sigma \dfrac{\partial}{\partial x}$ and

$$f(x,y) = \frac{-i|y|^\sigma}{\overline{Z(x,y)}\,\ln|Z(x,y)|} \in L^p(\mathbb{R}^2) \quad \text{for any } p < 2+\sigma$$

but $f \notin L^{p'}$ with $p' > 2+\sigma$. The function $u(x,y) = \ln|\ln Z(x,y)| \notin L^\infty$ and satisfies $Lu = f$. Equation $Lu = f$ has no bounded solutions.

Remark. $f \in L^p$ with $p > 2 + \sigma$ is optimal for obtaining bounded solutions.
For the standard vector field $\dfrac{\partial}{\partial y} - i|y|^\sigma \dfrac{\partial}{\partial x}$ and

$$f(x, y) = \frac{-i|y|^\sigma}{\overline{Z(x, y)} \ln |Z(x, y)|} \in L^p(\mathbb{R}^2) \quad \text{for any } p < 2 + \sigma$$

but $f \notin L^{p'}$ with $p' > 2 + \sigma$. The function $u(x, y) = \ln |\ln Z(x, y)| \notin L^\infty$ and satisfies $Lu = f$. Equation $Lu = f$ has no bounded solutions.

Remark. $f \in L^p$ with $p > 2 + \sigma$ is optimal for obtaining bounded solutions.
For the standard vector field $\dfrac{\partial}{\partial y} - i|y|^\sigma \dfrac{\partial}{\partial x}$ and

$$f(x, y) = \frac{-i|y|^\sigma}{\overline{Z(x,y)}\ \ln|Z(x,y)|} \in L^p(\mathbb{R}^2) \quad \text{for any } p < 2 + \sigma$$

but $f \notin L^{p'}$ with $p' > 2 + \sigma$. The function
$u(x, y) = \ln|\ln Z(x, y)| \notin L^\infty$ and satisfies $Lu = f$. Equation
$Lu = f$ has no bounded solutions.

Remark. $f \in L^p$ with $p > 2 + \sigma$ is optimal for obtaining bounded solutions.
For the standard vector field $\dfrac{\partial}{\partial y} - i|y|^\sigma \dfrac{\partial}{\partial x}$ and

$$f(x,y) = \frac{-i|y|^\sigma}{\overline{Z(x,y)}\,\ln|Z(x,y)|} \in L^p(\mathbb{R}^2) \quad \text{for any } p < 2 + \sigma$$

but $f \notin L^{p'}$ with $p' > 2 + \sigma$. The function $u(x,y) = \ln|\ln Z(x,y)| \notin L^\infty$ and satisfies $Lu = f$. Equation $Lu = f$ has no bounded solutions.

Remark. $f \in L^p$ with $p > 2 + \sigma$ is optimal for obtaining bounded solutions.
For the standard vector field $\dfrac{\partial}{\partial y} - i|y|^\sigma \dfrac{\partial}{\partial x}$ and

$$f(x,y) = \frac{-i|y|^\sigma}{\overline{Z(x,y)}\,\ln|Z(x,y)|} \in L^p(\mathbb{R}^2) \quad \text{for any } p < 2 + \sigma$$

but $f \notin L^{p'}$ with $p' > 2 + \sigma$. The function $u(x,y) = \ln|\ln Z(x,y)| \notin L^\infty$ and satisfies $Lu = f$. Equation $Lu = f$ has no bounded solutions.

Hölder continuity

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, and $Lu = f$, then $u \in C^\alpha(\Omega)$ with $\alpha = \dfrac{2 - q - \tau}{q}$

The proof uses the lemma

Lemma ([C. Campana, P. Dattori, A.M.])

For $z_1 \neq z_0$ and $\tau < 1$, we have

$$\int_{D(0,R)} \frac{d\xi\, d\eta}{|\eta|^\tau\, |\zeta - z_1|^q\, |\zeta - z_0|^q} \leq C(p,\tau)\, |z_1 - z_0|^{2-2q-2\tau}$$

Hölder continuity

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, and $Lu = f$, then $u \in C^\alpha(\Omega)$ with
$$\alpha = \frac{2 - q - \tau}{q}$$

The proof uses the lemma

Lemma ([C. Campana, P. Dattori, A.M.])

For $z_1 \neq z_0$ and $\tau < 1$, we have

$$\int_{D(0,R)} \frac{d\xi\, d\eta}{|\eta|^\tau\, |\zeta - z_1|^q\, |\zeta - z_0|^q} \leq C(p, \tau)\, |z_1 - z_0|^{2-2q-2\tau}$$

Hölder continuity

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, and $Lu = f$, then $u \in C^\alpha(\Omega)$ with
$$\alpha = \frac{2 - q - \tau}{q}$$

The proof uses the lemma

Lemma ([C. Campana, P. Dattori, A.M.])

For $z_1 \neq z_0$ and $\tau < 1$, we have

$$\int_{D(0,R)} \frac{d\xi\, d\eta}{|\eta|^\tau\, |\zeta - z_1|^q\, |\zeta - z_0|^q} \leq C(p, \tau)\, |z_1 - z_0|^{2-2q-2\tau}$$

Hölder continuity

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, and $Lu = f$, then $u \in C^\alpha(\Omega)$ with
$$\alpha = \frac{2 - q - \tau}{q}$$

The proof uses the lemma

Lemma ([C. Campana, P. Dattori, A.M.])

For $z_1 \neq z_0$ and $\tau < 1$, we have

$$\int_{D(0,R)} \frac{d\xi\, d\eta}{|\eta|^\tau\, |\zeta - z_1|^q\, |\zeta - z_0|^q} \leq C(p,\tau)\, |z_1 - z_0|^{2-2q-2\tau}$$

Hölder continuity

Theorem ([C. Campana, P. Dattori, A.M.])

If $f \in L^p(\Omega)$ with $p > 2 + \sigma$, and $Lu = f$, then $u \in C^\alpha(\Omega)$ with
$$\alpha = \frac{2 - q - \tau}{q}$$

The proof uses the lemma

Lemma ([C. Campana, P. Dattori, A.M.])

For $z_1 \neq z_0$ and $\tau < 1$, we have

$$\int_{D(0,R)} \frac{d\xi\, d\eta}{|\eta|^\tau\, |\zeta - z_1|^q\, |\zeta - z_0|^q} \leq C(p, \tau)\, |z_1 - z_0|^{2-2q-2\tau}$$

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi, \eta)|\, d\xi d\eta}{|Z(\xi, \eta) - z_1|\, |Z(\xi, \eta) - z_0|}$$

$$\leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi, \eta) - z_1|^q\, |Z(\xi, \eta) - z_0|^q}$$

Case $Z(x, y) = x + \dfrac{y|y|^\sigma}{\sigma + 1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0, R)$ and $s = \xi$, $t = \dfrac{y|y|^\sigma}{\sigma + 1}$ It follows from the lemma that

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi,\eta)|\, d\xi d\eta}{|Z(\xi,\eta) - z_1|\ |Z(\xi,\eta) - z_0|}$$

$$\leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi,\eta) - z_1|^q\ |Z(\xi,\eta) - z_0|^q}$$

Case $Z(x,y) = x + \frac{y|y|^\sigma}{\sigma + 1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0, R)$ and $s = \xi$, $t = \frac{y|y|^\sigma}{\sigma + 1}$ It follows from the lemma that

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi, \eta)|\, d\xi d\eta}{|Z(\xi, \eta) - z_1|\ |Z(\xi, \eta) - z_0|}$$

$$\leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi, \eta) - z_1|^q\ |Z(\xi, \eta) - z_0|^q}$$

Case $Z(x, y) = x + \frac{y|y|^\sigma}{\sigma + 1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0, R)$ and $s = \xi$, $t = \frac{y|y|^\sigma}{\sigma + 1}$ It follows from the lemma that

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi,\eta)|\, d\xi d\eta}{|Z(\xi,\eta) - z_1|\ |Z(\xi,\eta) - z_0|}$$

$$\leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi,\eta) - z_1|^q\ |Z(\xi,\eta) - z_0|^q}$$

Case $Z(x,y) = x + \dfrac{y|y|^\sigma}{\sigma+1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0,R)$ and $s = \xi$, $t = \dfrac{y|y|^\sigma}{\sigma+1}$ It follows from the lemma that

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi, \eta)|\, d\xi d\eta}{|Z(\xi, \eta) - z_1|\ |Z(\xi, \eta) - z_0|}$$

$$\leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi, \eta) - z_1|^q\ |Z(\xi, \eta) - z_0|^q}$$

Case $Z(x, y) = x + \dfrac{y|y|^\sigma}{\sigma + 1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0, R)$ and $s = \xi$, $t = \dfrac{y|y|^\sigma}{\sigma + 1}$ It follows from the lemma that

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \le \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi, \eta)|\, d\xi d\eta}{|Z(\xi, \eta) - z_1|\; |Z(\xi, \eta) - z_0|}$$

$$\le \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi, \eta) - z_1|^q \; |Z(\xi, \eta) - z_0|^q}$$

Case $Z(x, y) = x + \dfrac{y|y|^\sigma}{\sigma + 1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0, R)$ and $s = \xi$, $t = \dfrac{y|y|^\sigma}{\sigma + 1}$ It follows from the lemma that

Proof of Theorem

Set $z_1 = Z(x_1, y_1)$ and $z_0 = Z(x_0, y_0)$.

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{|z_1 - z_0|}{\pi} \int_\Omega \frac{|f(\xi, \eta)|\, d\xi d\eta}{|Z(\xi, \eta) - z_1|\ |Z(\xi, \eta) - z_0|}$$

$$\leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}}$$

with

$$H = \int_\Omega \frac{d\xi d\eta}{|Z(\xi, \eta) - z_1|^q\ |Z(\xi, \eta) - z_0|^q}$$

Case $Z(x, y) = x + \dfrac{y|y|^\sigma}{\sigma + 1}$. Let $R > 0$ s.t. $Z(\Omega) \subset D(0, R)$ and $s = \xi$, $t = \dfrac{y|y|^\sigma}{\sigma + 1}$ It follows from the lemma that

$$H \leq \frac{1}{(1+\sigma)^\tau} \int_{D(0,R)} \frac{ds\, dt}{|t|^\tau \, |\zeta - z_1|^q \, |\zeta - z_0|^q}$$

$$\leq C(q,\tau)|z_1 - z_0|^{2-2q-\tau}$$

and

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}} \leq C(p,\tau)|z_1 - z_0|^\alpha$$

with $\alpha = \dfrac{2-q-\tau}{q}$.

The general case can be reduced to this special case as was done previously.

$$H \leq \frac{1}{(1+\sigma)^\tau} \int_{D(0,R)} \frac{ds\,dt}{|t|^\tau \, |\zeta - z_1|^q \, |\zeta - z_0|^q}$$

$$\leq C(q,\tau)|z_1 - z_0|^{2-2q-\tau}$$

and

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{||f||_p}{\pi}|z_1 - z_0| H^{\frac{1}{q}} \leq C(p,\tau)|z_1 - z_0|^\alpha$$

with $\alpha = \dfrac{2-q-\tau}{q}$.
The general case can be reduced to this special case as was done previously.

$$H \leq \frac{1}{(1+\sigma)^\tau} \int_{D(0,R)} \frac{ds\,dt}{|t|^\tau \, |\zeta - z_1|^q \, |\zeta - z_0|^q}$$

$$\leq C(q,\tau)|z_1 - z_0|^{2-2q-\tau}$$

and

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \; \leq \; \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}} \; \leq \; C(p,\tau)|z_1 - z_0|^\alpha$$

with $\alpha = \dfrac{2-q-\tau}{q}$.

The general case can be reduced to this special case as was done previously.

$$H \leq \frac{1}{(1+\sigma)^\tau} \int_{D(0,R)} \frac{ds\,dt}{|t|^\tau \, |\zeta - z_1|^q \, |\zeta - z_0|^q}$$

$$\leq C(q,\tau)|z_1 - z_0|^{2-2q-\tau}$$

and

$$|T_Z f(x_1, y_1) - T_Z f(x_0, y_0)| \leq \frac{||f||_p}{\pi} |z_1 - z_0| H^{\frac{1}{q}} \leq C(p,\tau)|z_1 - z_0|^\alpha$$

with $\alpha = \dfrac{2-q-\tau}{q}$.
The general case can be reduced to this special case as was done previously.

Semilinear Equation

The regularity of solutions of $u_y = f(x, y, u, u_x)$ was studied by many authors using the FBI transform. Including: [Adwan, Berhanu](2012); [Asano] (1995); [Baouendi-Goulaouic-Treves](1985); [Berhanu] (2009), [Lerner, Morimoto, Xu] (2008). The last chapter of Treves book "Hypo-analytic Structures" deals with nonlinear structures.

However, in all these works, the function $f(x, y, \zeta_0, \zeta_1)$ is assumed to be holomorphic in (ζ_0, ζ_1).

A paper by Hounie and Santiago (1996) considers the equation $Lu = f(x, u)$ with L satisfies $(\mathcal{P})$. They prove in particular existence of L^p solutions if $f \in L^p$.

Semilinear Equation

The regularity of solutions of $u_y = f(x, y, u, u_x)$ was studied by many authors using the FBI transform. Including: [Adwan, Berhanu](2012); [Asano] (1995); [Baouendi-Goulaouic-Treves](1985); [Berhanu] (2009), [Lerner, Morimoto, Xu] (2008). The last chapter of Treves book "Hypo-analytic Structures" deals with nonlinear structures.

However, in all these works, the function $f(x, y, \zeta_0, \zeta_1)$ is assumed to be holomorphic in (ζ_0, ζ_1).

A paper by Hounie and Santiago (1996) considers the equation $Lu = f(x, u)$ with L satisfies $(\mathcal{P})$. They prove in particular existence of L^p solutions if $f \in L^p$.

Semilinear Equation

The regularity of solutions of $u_y = f(x, y, u, u_x)$ was studied by many authors using the FBI transform. Including: [Adwan, Berhanu](2012); [Asano] (1995); [Baouendi-Goulaouic-Treves](1985); [Berhanu] (2009), [Lerner, Morimoto, Xu] (2008). The last chapter of Treves book "Hypo-analytic Structures" deals with nonlinear structures.

However, in all these works, the function $f(x, y, \zeta_0, \zeta_1)$ is assumed to be holomorphic in (ζ_0, ζ_1).

A paper by Hounie and Santiago (1996) considers the equation $Lu = f(x, u)$ with L satisfies $(\mathcal{P})$. They prove in particular existence of L^p solutions if $f \in L^p$.

Semilinear Equation

The regularity of solutions of $u_y = f(x, y, u, u_x)$ was studied by many authors using the FBI transform. Including: [Adwan, Berhanu](2012); [Asano] (1995); [Baouendi-Goulaouic-Treves](1985); [Berhanu] (2009), [Lerner, Morimoto, Xu] (2008). The last chapter of Treves book "Hypo-analytic Structures" deals with nonlinear structures.

However, in all these works, the function $f(x, y, \zeta_0, \zeta_1)$ is assumed to be holomorphic in (ζ_0, ζ_1).

A paper by Hounie and Santiago (1996) considers the equation $Lu = f(x, u)$ with L satisfies $(\mathcal{P})$. They prove in particular existence of L^p solutions if $f \in L^p$.

Semilinear Equation

The regularity of solutions of $u_y = f(x, y, u, u_x)$ was studied by many authors using the FBI transform. Including: [Adwan, Berhanu](2012); [Asano] (1995); [Baouendi-Goulaouic-Treves](1985); [Berhanu] (2009), [Lerner, Morimoto, Xu] (2008). The last chapter of Treves book "Hypo-analytic Structures" deals with nonlinear structures.

However, in all these works, the function $f(x, y, \zeta_0, \zeta_1)$ is assumed to be holomorphic in (ζ_0, ζ_1).

A paper by Hounie and Santiago (1996) considers the equation $Lu = f(x, u)$ with L satisfies $(\mathcal{P})$. They prove in particular existence of L^p solutions if $f \in L^p$.

Semilinear Equation

The regularity of solutions of $u_y = f(x, y, u, u_x)$ was studied by many authors using the FBI transform. Including: [Adwan, Berhanu](2012); [Asano] (1995); [Baouendi-Goulaouic-Treves](1985); [Berhanu] (2009), [Lerner, Morimoto, Xu] (2008). The last chapter of Treves book "Hypo-analytic Structures" deals with nonlinear structures.

However, in all these works, the function $f(x, y, \zeta_0, \zeta_1)$ is assumed to be holomorphic in (ζ_0, ζ_1).

A paper by Hounie and Santiago (1996) considers the equation $Lu = f(x, u)$ with L satisfies $(\mathcal{P})$. They prove in particular existence of L^p solutions if $f \in L^p$.

Let $F(x, y, \zeta)$ be a function defined on $\Omega \times \mathbb{C}$ such that

- For every $\zeta \in \mathbb{C}$, $F(., \zeta) \in L^p(\Omega)$ with $p > 2 + \sigma$;
- There exists $\psi(x, y) \in L^p(\Omega)$ and $\beta \in (0,\ 1]$ s.t.

$$|F(x, y, \zeta_1) - F(x, y, \zeta_0)| \ \leq \ \psi(x, y)\, |\zeta_1 - \zeta_0|^{\beta}$$

Let $F(x, y, \zeta)$ be a function defined on $\Omega \times \mathbb{C}$ such that

- For every $\zeta \in \mathbb{C}$, $F(., \zeta) \in L^p(\Omega)$ with $p > 2 + \sigma$;
- There exists $\psi(x, y) \in L^p(\Omega)$ and $\beta \in (0,\ 1]$ s.t.

$$|F(x, y, \zeta_1) - F(x, y, \zeta_0)| \;\leq\; \psi(x, y)\, |\zeta_1 - \zeta_0|^{\beta}$$

Let $F(x, y, \zeta)$ be a function defined on $\Omega \times \mathbb{C}$ such that

- For every $\zeta \in \mathbb{C}$, $F(., \zeta) \in L^p(\Omega)$ with $p > 2 + \sigma$;
- There exists $\psi(x, y) \in L^p(\Omega)$ and $\beta \in (0,\ 1]$ s.t.

$$|F(x, y, \zeta_1) - F(x, y, \zeta_0)| \leq \psi(x, y)\, |\zeta_1 - \zeta_0|^{\beta}$$

Let $F(x, y, \zeta)$ be a function defined on $\Omega \times \mathbb{C}$ such that

- For every $\zeta \in \mathbb{C}$, $F(., \zeta) \in L^p(\Omega)$ with $p > 2 + \sigma$;
- There exists $\psi(x, y) \in L^p(\Omega)$ and $\beta \in (0,\ 1]$ s.t.

$$|F(x, y, \zeta_1) - F(x, y, \zeta_0)| \ \leq \ \psi(x, y)\, |\zeta_1 - \zeta_0|^{\beta}$$

Semilinear equation

Theorem ([C. Campana, P. Dattori, A.M.])

Let $F(x, y, u)$ *be as above,* $\alpha = \dfrac{2 - q - \tau}{q}$, $\tau = \dfrac{\sigma}{\sigma + 1}$. *Then*

- *If* $\beta < 1$, *equation* $Lu = F(x, y, u)$ *has a solution in* $C^\alpha(\Omega)$;
- *If* $\beta = 1$, *then for every* $(x, y) \in \Omega$, *there is an open neighborhood* $U \subset \Omega$ *such that equation* $Lu = F(x, y, u)$ *has a solution* $u \in C^\alpha(U)$.

Semilinear equation

Theorem ([C. Campana, P. Dattori, A.M.])

Let $F(x, y, u)$ be as above, $\alpha = \dfrac{2 - q - \tau}{q}$, $\tau = \dfrac{\sigma}{\sigma + 1}$. Then

- *If $\beta < 1$, equation $Lu = F(x, y, u)$ has a solution in $C^{\alpha}(\Omega)$;*
- *If $\beta = 1$, then for every $(x, y) \in \Omega$, there is an open neighborhood $U \subset \Omega$ such that equation $Lu = F(x, y, u)$ has a solution $u \in C^{\alpha}(U)$.*

Semilinear equation

Theorem ([C. Campana, P. Dattori, A.M.])

Let $F(x, y, u)$ be as above, $\alpha = \dfrac{2-q-\tau}{q}$, $\tau = \dfrac{\sigma}{\sigma+1}$. Then

- *If $\beta < 1$, equation $Lu = F(x, y, u)$ has a solution in $C^\alpha(\Omega)$;*
- *If $\beta = 1$, then for every $(x, y) \in \Omega$, there is an open neighborhood $U \subset \Omega$ such that equation $Lu = F(x, y, u)$ has a solution $u \in C^\alpha(U)$.*

Semilinear equation

Theorem ([C. Campana, P. Dattori, A.M.])

Let $F(x, y, u)$ *be as above,* $\alpha = \dfrac{2 - q - \tau}{q}$, $\tau = \dfrac{\sigma}{\sigma + 1}$. *Then*

- *If* $\beta < 1$, *equation* $Lu = F(x, y, u)$ *has a solution in* $C^\alpha(\Omega)$;
- *If* $\beta = 1$, *then for every* $(x, y) \in \Omega$, *there is an open neighborhood* $U \subset \Omega$ *such that equation* $Lu = F(x, y, u)$ *has a solution* $u \in C^\alpha(U)$.

Semilinear equation

Theorem ([C. Campana, P. Dattori, A.M.])

Let $F(x,y,u)$ be as above, $\alpha = \dfrac{2-q-\tau}{q}$, $\tau = \dfrac{\sigma}{\sigma+1}$. Then

- *If $\beta < 1$, equation $Lu = F(x,y,u)$ has a solution in $C^\alpha(\Omega)$;*
- *If $\beta = 1$, then for every $(x,y) \in \Omega$, there is an open neighborhood $U \subset \Omega$ such that equation $Lu = F(x,y,u)$ has a solution $u \in C^\alpha(U)$.*

Proof

For $N > 0$, let

$$C_N(\overline{\Omega}) = \{u \in \mathbb{C}(\overline{\Omega});\ ||u||_\infty \leq N\}$$

Let $M(p, \sigma, \Omega)$ found earlier such that

$$||T_Z f||_\infty \leq M(p, \sigma, \Omega)||f||_p \quad \forall f \in L^p(\Omega), \quad p > 2 + \sigma$$

Case $\beta < 1$. Let N be large enough so that

$$M(p, \sigma, \Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right) \leq N\,.$$

Proof

For $N > 0$, let

$$C_N(\overline{\Omega}) = \{u \in \mathbb{C}(\overline{\Omega});\ ||u||_\infty \leq N\}$$

Let $M(p, \sigma, \Omega)$ found earlier such that

$$||T_Z f||_\infty \leq M(p, \sigma, \Omega)||f||_p \quad \forall f \in L^p(\Omega), \quad p > 2 + \sigma$$

Case $\beta < 1$. Let N be large enough so that

$$M(p, \sigma, \Omega)\left(||\psi||_p N^\beta + ||F(., 0)||_p\right) \leq N\,.$$

Proof

For $N > 0$, let

$$C_N(\overline{\Omega}) = \{u \in \mathbb{C}(\overline{\Omega});\ ||u||_\infty \leq N\}$$

Let $M(p, \sigma, \Omega)$ found earlier such that

$$||T_Z f||_\infty \leq M(p, \sigma, \Omega)||f||_p \quad \forall f \in L^p(\Omega), \quad p > 2 + \sigma$$

Case $\beta < 1$. Let N be large enough so that

$$M(p, \sigma, \Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right) \leq N .$$

Proof

For $N > 0$, let

$$C_N(\overline{\Omega}) = \{u \in \mathbb{C}(\overline{\Omega});\ ||u||_\infty \leq N\}$$

Let $M(p, \sigma, \Omega)$ found earlier such that

$$||T_Z f||_\infty \leq M(p, \sigma, \Omega)||f||_p \quad \forall f \in L^p(\Omega), \quad p > 2 + \sigma$$

Case $\beta < 1$. Let N be large enough so that

$$M(p, \sigma, \Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right) \leq N\,.$$

Proof

For $N > 0$, let

$$C_N(\overline{\Omega}) = \{u \in \mathbb{C}(\overline{\Omega});\ ||u||_\infty \leq N\}$$

Let $M(p, \sigma, \Omega)$ found earlier such that

$$||T_Z f||_\infty \leq M(p, \sigma, \Omega)||f||_p \quad \forall f \in L^p(\Omega), \quad p > 2 + \sigma$$

Case $\beta < 1$. Let N be large enough so that

$$M(p, \sigma, \Omega) \left(||\psi||_p N^\beta + ||F(., 0)||_p \right) \leq N\,.$$

Proof

For $N > 0$, let

$$C_N(\overline{\Omega}) = \{u \in \mathbb{C}(\overline{\Omega});\ ||u||_\infty \leq N\}$$

Let $M(p, \sigma, \Omega)$ found earlier such that

$$||T_Z f||_\infty \leq M(p, \sigma, \Omega)||f||_p \quad \forall f \in L^p(\Omega), \quad p > 2 + \sigma$$

Case $\beta < 1$. Let N be large enough so that

$$M(p, \sigma, \Omega)\left(||\psi||_p N^\beta + ||F(., 0)||_p\right) \leq N\,.$$

Let $\mathbf{P} : \; C_N(\overline{\Omega}) \longrightarrow C_N(\overline{\Omega})$:

$$\mathbf{P}u(x,y) = T_Z F(x,y,u(x,y)) \,.$$

$\mathbf{P}$ is well defined since

$$\begin{aligned}
|\mathbf{P}u(x,y)| &\leq M(p,\sigma,\Omega)||F(x,y,u)||_p \\
&\leq M(p,\sigma,\Omega)\left(||F(x,y,u) - F(x,y,0)||_p + ||F(x,y,0)||_p\right) \\
&\leq M(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(x,y,0)||_p\right) \leq N
\end{aligned}$$

Also the continuity of $\mathbf{P}u$ follows from that of the integral operator $T_Z F$:

$$\begin{aligned}
|\mathbf{P}u(m_1) - \mathbf{P}u(m_0)| &\leq C(p,\sigma,\Omega)||F(.,u)||_p \, |Z(m_1) - Z(m_0)|^\alpha \\
&\leq C(p,\sigma,\Omega)N \, |Z(m_1) - Z(m_0)|^\alpha
\end{aligned}$$

Let $\mathbf{P}: C_N(\overline{\Omega}) \longrightarrow C_N(\overline{\Omega})$:

$$\mathbf{P}u(x,y) = T_Z F(x,y,u(x,y))\,.$$

$\mathbf{P}$ is well defined since

$$\begin{aligned}|\mathbf{P}u(x,y)| &\le M(p,\sigma,\Omega)||F(x,y,u)||_p \\ &\le M(p,\sigma,\Omega)\left(||F(x,y,u)-F(x,y,0)||_p + ||F(x,y,0)||_p\right) \\ &\le M(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(x,y,0)||_p\right) \le N\end{aligned}$$

Also the continuity of $\mathbf{P}u$ follows from that of the integral operator $T_Z F$:

$$\begin{aligned}|\mathbf{P}u(m_1)-\mathbf{P}u(m_0)| &\le C(p,\sigma,\Omega)||F(.,u)||_p\,|Z(m_1)-Z(m_0)|^\alpha \\ &\le C(p,\sigma,\Omega)N\,|Z(m_1)-Z(m_0)|^\alpha\end{aligned}$$

Let $\mathbf{P}: \; C_N(\overline{\Omega}) \longrightarrow C_N(\overline{\Omega})$:

$$\mathbf{P}u(x,y) = T_Z F(x,y,u(x,y))\,.$$

P is well defined since

$$\begin{aligned}|\mathbf{P}u(x,y)| &\le M(p,\sigma,\Omega)||F(x,y,u)||_p \\ &\le M(p,\sigma,\Omega)\left(||F(x,y,u)-F(x,y,0)||_p + ||F(x,y,0)||_p\right) \\ &\le M(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(x,y,0)||_p\right) \le N\end{aligned}$$

Also the continuity of $\mathbf{P}u$ follows from that of the integral operator $T_Z F$:

$$\begin{aligned}|\mathbf{P}u(m_1) - \mathbf{P}u(m_0)| &\le C(p,\sigma,\Omega)||F(.,u)||_p\,|Z(m_1)-Z(m_0)|^\alpha \\ &\le C(p,\sigma,\Omega)N\,|Z(m_1)-Z(m_0)|^\alpha\end{aligned}$$

Let $\mathbf{P}: C_N(\overline{\Omega}) \longrightarrow C_N(\overline{\Omega})$:

$$\mathbf{P}u(x,y) = T_Z F(x,y,u(x,y))\,.$$

$\mathbf{P}$ is well defined since

$$\begin{aligned}|\mathbf{P}u(x,y)| &\leq M(p,\sigma,\Omega)||F(x,y,u)||_p \\ &\leq M(p,\sigma,\Omega)\,(||F(x,y,u)-F(x,y,0)||_p + ||F(x,y,0)||_p) \\ &\leq M(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(x,y,0)||_p\right) \leq N\end{aligned}$$

Also the continuity of $\mathbf{P}u$ follows from that of the integral operator $T_Z F$:

$$\begin{aligned}|\mathbf{P}u(m_1)-\mathbf{P}u(m_0)| &\leq C(p,\sigma,\Omega)||F(.,u)||_p\,|Z(m_1)-Z(m_0)|^\alpha \\ &\leq C(p,\sigma,\Omega)N\,|Z(m_1)-Z(m_0)|^\alpha\end{aligned}$$

Let $\mathbf{P} : \; C_N(\overline{\Omega}) \;\longrightarrow\; C_N(\overline{\Omega})$:

$$\mathbf{P}u(x,y) = T_Z F(x,y,u(x,y))\,.$$

P is well defined since

$$\begin{aligned}|\mathbf{P}u(x,y)| &\le M(p,\sigma,\Omega)||F(x,y,u)||_p \\ &\le M(p,\sigma,\Omega)\,(||F(x,y,u)-F(x,y,0)||_p + ||F(x,y,0)||_p) \\ &\le M(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(x,y,0)||_p\right) \le N\end{aligned}$$

Also the continuity of $\mathbf{P}u$ follows from that of the integral operator $T_Z F$:

$$\begin{aligned}|\mathbf{P}u(m_1) - \mathbf{P}u(m_0)| &\le C(p,\sigma,\Omega)||F(.,u)||_p\,|Z(m_1)-Z(m_0)|^\alpha \\ &\le C(p,\sigma,\Omega)N\,|Z(m_1)-Z(m_0)|^\alpha\end{aligned}$$

Let $\mathbf{P} : \; C_N(\overline{\Omega}) \; \longrightarrow \; C_N(\overline{\Omega})$:

$$\mathbf{P}u(x,y) = T_Z F(x,y,u(x,y))\,.$$

$\mathbf{P}$ is well defined since

$$\begin{aligned}
|\mathbf{P}u(x,y)| &\leq M(p,\sigma,\Omega)||F(x,y,u)||_p \\
&\leq M(p,\sigma,\Omega)\,(||F(x,y,u) - F(x,y,0)||_p + ||F(x,y,0)||_p) \\
&\leq M(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(x,y,0)||_p\right) \leq N
\end{aligned}$$

Also the continuity of $\mathbf{P}u$ follows from that of the integral operator $T_Z F$:

$$\begin{aligned}
|\mathbf{P}u(m_1) - \mathbf{P}u(m_0)| &\leq C(p,\sigma,\Omega)||F(.,u)||_p\,|Z(m_1) - Z(m_0)|^\alpha \\
&\leq C(p,\sigma,\Omega)N\,|Z(m_1) - Z(m_0)|^\alpha
\end{aligned}$$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) : \ |u(m_1) - u(m_0)| \leq C\,|Z(m_1) - Z(m_0)|^\alpha\}.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela). We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned}|\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)|\\ &\leq M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p\\ &\leq M(p,\sigma,\Omega)||\psi||_p||u-v||_\infty^\beta\end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus $Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) :\ |u(m_1) - u(m_0)| \leq C\,|Z(m_1) - Z(m_0)|^\alpha\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela). We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned}|\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)|\\ &\leq M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p\\ &\leq M(p,\sigma,\Omega)||\psi||_p||u-v||_\infty^\beta\end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus $Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^{\beta} + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) :\ |u(m_1) - u(m_0)| \leq C\,|Z(m_1) - Z(m_0)|^{\alpha}\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela).
We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned} |\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)| \\ &\leq M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p \\ &\leq M(p,\sigma,\Omega)||\psi||_p||u - v||_{\infty}^{\beta} \end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus $Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) : \ |u(m_1) - u(m_0)| \le C\,|Z(m_1) - Z(m_0)|^\alpha\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela).
We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned} |\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)| \\ &\le M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p \\ &\le M(p,\sigma,\Omega)||\psi||_p||u-v||_\infty^\beta \end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus $Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) : \ |u(m_1) - u(m_0)| \leq C\,|Z(m_1) - Z(m_0)|^\alpha\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela).
We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned}
|\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)| \\
&\leq M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p \\
&\leq M(p,\sigma,\Omega)||\psi||_p||u - v||_\infty^\beta
\end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus
$Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) : \ |u(m_1) - u(m_0)| \leq C\,|Z(m_1) - Z(m_0)|^\alpha\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela).
We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned} |\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)| \\ &\leq M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p \\ &\leq M(p,\sigma,\Omega)||\psi||_p||u-v||_\infty^\beta \end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus
$Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p,\sigma,\Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) :\ |u(m_1) - u(m_0)| \le C\,|Z(m_1) - Z(m_0)|^\alpha\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela).
We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned} |\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)| \\ &\le M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p \\ &\le M(p,\sigma,\Omega)||\psi||_p||u - v||_\infty^\beta \end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus
$Lu = LT_Z F(x,y,u) = F(x,y,u)$

Let $C = C(p, \sigma, \Omega)\left(||\psi||_p N^\beta + ||F(.,0)||_p\right)$. and

$$\Lambda_{N,C} = \{u \in C_N(\overline{\Omega}) : \ |u(m_1) - u(m_0)| \leq C\,|Z(m_1) - Z(m_0)|^\alpha\}\,.$$

$\Lambda_{N,C}$ is a convex subset of $C_N(\overline{\Omega})$ and compact (Ascoli-Arzela).
We have $\mathbf{P}(C_N(\overline{\Omega})) \subset \Lambda_{N,C}$ and

$$\begin{aligned}
|\mathbf{P}u(m) - \mathbf{P}v(m)| &= |T_Z F(m,u) - T_Z F(m,v)| \\
&\leq M(p,\sigma,\Omega)||F(.,u) - F(.,v)||_p \\
&\leq M(p,\sigma,\Omega)||\psi||_p||u - v||_\infty^\beta
\end{aligned}$$

$\mathbf{P} : \Lambda_{N,C} \longrightarrow \Lambda_{N,C}$ is continuous. By Schauder fixed point theorem, there is $u \in \Lambda_{N,C}$ satisfying $u = \mathbf{P}u$ .Thus
$Lu = LT_Z F(x,y,u) = F(x,y,u)$