II. Similarity Principle; Boundary Value Problems; Applications

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Similarity Principle for $\overline{\partial}$

Theorem

Let A, $B \in L^p(\Omega)$ with $p > 2$. If w solves

$$\frac{\partial w}{\partial \overline{z}} = Aw + B\overline{w}\,, \tag{0.1}$$

then there exist a holomorphic function h in Ω and $s \in C^\alpha(\Omega)$, with $\alpha = \dfrac{p-2}{p}$ such that

$$u(z) = h(z)e^{s(z)}\,. \tag{0.2}$$

Conversely, for any given holomorphic function h in Ω, there exists a Hölder continuous function $s \in C^\alpha(\Omega)$ such that u given by (2) satisfies (1)

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In 2000, Berhanu-Hounie-Santiago generalized a weak version of the principle to vector fields satisfying condition $\mathcal{P}$.They used the result to show uniqueness of the Cauchy problem for vector fields.

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For the class of vector under consideration. We prove a strong version of the principle.

Recall: $Z : \overline{\Omega} \longrightarrow Z(\overline{\Omega}) \subset \mathbb{C}$ is a C^1 homeomorphism,

$$\Sigma = \{p \in \overline{\Omega} : \ \mathrm{Im}(Z_x \overline{Z}_y)(p) = 0\}$$

is a C^1 1-dimensional submanifold of $\overline{\Omega}$.
Near each point p of Σ, there are coordinates such that

$$Z(x,t) = x + i\phi(x,t)$$

and $|\phi(x,t) - \phi(x,0)|^{\tau} \leq C\,|\phi_t(x,t)|$ with $\tau < 1$ and $C > 0$.

$$L = Z_x(x,y)\frac{\partial}{\partial y} - Z_y(x,y)\frac{\partial}{\partial x}$$

$$T_Z f(x,y) = \frac{1}{2\pi i}\int_{\Omega} \frac{f(\xi,\eta)\,d\xi d\eta}{Z(\xi,\eta) - Z(x,y)}$$

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If $f \in L^p(\Omega)$, with $p > 2 + \sigma$, then

$$||T_Z f||_\infty \leq C(p, \sigma, \Omega)||f||_p\,;$$

$$LT_Z f = f$$

$$|T_Z f(p) - T_Z f(q)| \leq C(p, \sigma, \Omega)||f||_p\,|Z(p) - Z(q)|^\alpha$$

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A semilinear equation

Let $G(x, y, \zeta) \in C^0(\overline{\Omega} \times \mathbb{C}) \cap L^\infty(\overline{\Omega} \times \mathbb{C})$.

Theorem (C. Campana, P. Dattori, A.M)

Let G be as above and A, $B \in L^p(\overline{\Omega})$ with $p > 2 + \sigma$. Then, equation

$$Lu = A(x, y) + B(x, y)G(x, y, u)$$

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Proof

Let $\mathbf{P} : C^0(\overline{\Omega}) \longrightarrow C^0(\overline{\Omega})$ given by

$$\mathbf{P}u(x,y) = T_Z\left(A(x,y) + B(x,y)G(x,y,u)\right) .$$

Since $A(x,y) + B(x,y)G(x,y,u) \in L^p(\Omega)$, then

$$\begin{aligned} ||\mathbf{P}u||_\infty &\leq M(p,\sigma,\Omega)||A+BG||_p \\ &\leq M(||A||_p + ||G||_\infty||B||_p) = D \end{aligned}$$

and

$$\begin{aligned} |\mathbf{P}u(m_1) - \mathbf{P}u(m_2)| &\leq |T_Z(A+BG)(m_1) - T_Z(A+BG)(m_2)| \\ &\leq C\left(||A||_p + ||G||_\infty||B||_p\right)\, |Z(m_1) - Z(m_2)|^\alpha \end{aligned}$$

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Let $E = C(||A||_p + ||G||_\infty ||B||_p)$ and $\Lambda_{D,E}$ be the set of function $v \in C^0(\overline{\Omega})$ such that

$$||v||_\infty \le D \quad \text{and} \quad |v(m_1) - v(m_2)| \le E\,|Z(m_1) - Z(m_2)|^\alpha$$

Then $\mathbf{P}(\Lambda_{D,E}) \subset \Lambda_{D,E}$ and $\mathbf{P}$ is a continuous operator:Since G is uniformly continuous on the compact

$$\overline{\Omega} \times \{\zeta \in \mathbb{C};\ |\zeta| \le D\}$$

then for $\epsilon > 0$, there is $\delta > 0$, s.t.

$$|G(m,\zeta_1) - G(m,\zeta_2)| \le \frac{\epsilon}{M(||B||_p + 1)} \quad \text{for} \quad |\zeta_1 - \zeta_2| \le \delta$$

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$$\begin{aligned} ||\mathbf{P}u - \mathbf{P}v||_\infty &= ||T_Z(B(G(u) - G(v)))||_\infty \\ &\leq M\,||B(G(u) - G(v))||_p \\ &\leq M||B||_p\,||G(u) - G(v)||_\infty \\ &\leq \frac{M\,||B||_p}{M(||B||_p + 1)}\,\epsilon \leq \epsilon \end{aligned}$$

Shauder Fixed Point Theorem implies that $\mathbf{P}$ has a fixed point u in $\Lambda D, E$ and so

$$Lu = A(x, y) + B(x, y)G(x, y, u)$$

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Similarity Principle

Theorem (C. Campana, P. Dattori, A.M)

Let $a,\ b\ \in, L^p(\Omega)$ *with* $p > 2+\sigma$.

- *If* $u \in L^\infty(\Omega)$ *satisfies*

$$Lu = au + b\overline{u} \tag{0.3}$$

  then there is a holomorphic function H *in* $Z(\Omega) \subset \mathbb{C}$ *and* $s \in C^\alpha(\overline{\Omega})$ *such that*

$$u(x,y) = H(Z(x,y))e^{s(x,y)} \tag{0.4}$$

- *If* H *is a holomorphic function defined in* $Z(\Omega)$, *then there exists* $s \in C^\alpha(\overline{\Omega})$ *such that* u *given by (4) satisfies (3)*

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Second order elliptic equations with degeneracies

$$P = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y}$$

For simplicity, assume that the coefficients are real analytic. Suppose that $AC - B^2 \geq 0$ and $A = 1$. The degeneracy set $\{C - B^2 = 0\}$ is then an analytic variety of dimension 1.Let

$$L = \partial_x + \beta\partial_y\,, \quad \text{with} \quad \beta = B + iS = B + i\sqrt{C - B^2}$$

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$$N = \frac{R_s \times R_t}{|R_s \times R_t|}$$

The Gaussian curvature

$$K = \frac{eg - f^2}{EG - F^2}$$

An *infinitesimal bending* of S of class C^k is a deformation $S_\sigma \subset \mathbb{R}^3$, with $\sigma \in \mathbb{R}^3$ with position vector

$$R_\sigma(s,t) = R(s,t) + \sigma U(s,t)$$

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$$dR_\sigma^2 = dR^2 + O(\sigma^2) \quad \text{as } \sigma \to 0$$

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Existence of nontrivial bending fields

Assume $K > 0$ on $\overline{\Omega}$ except at finitely many planar points $p_1, \cdots, p_l \in \Omega$.

Theorem ([A.M] (2013))

For every $k \in \mathbb{Z}^+$, the surface S has nontrivial infinitesimal bendings $U : \overline{\Omega} \longrightarrow \mathbb{R}^3$ of class C^k.

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Reduction to solvability of vector fields

The (complex) asymptotic directions are given by

$$\lambda^2 + 2f\lambda + eg = 0\,; \quad \lambda = -f + i\sqrt{eg - f^2} \in \mathbb{R} + i\mathbb{R}^+\,.$$

The vector field

$$L = g(s,t)\frac{\partial}{\partial s} + \lambda(s,t)\frac{\partial}{\partial t}$$

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Solvability of $Lw = Aw + B\overline{w}$

First find a global first integral of L

Proposition ([A.M.] (2013))

There exists $Z : \overline{\Omega} \longrightarrow Z(\overline{\Omega}) \subset \mathbb{C}$ homeomorphism

- $Z \in C^\infty(\overline{\Omega} \backslash \{p_1, \cdots, p_l\})$
- *For every j, there is $\mu_j > 0$ and local polar coordinates (r, θ) centered at p_j such that*

$$Z(r,\theta) = Z(0,0) + r^{\mu_j} e^{i\theta} + O(r^{2\mu_j})$$

- $LZ = 0$ *on* $\overline{\Omega}$

Use the first integral to convert into a Bers-Vekua type equation with singular points

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$$\frac{\partial W}{\partial \overline{Z}} = \frac{P(Z)}{\prod_{j=1}^{l}(Z - Z_j)} W + \frac{Q(Z)}{\prod_{j=1}^{l}(Z - Z_j)} \overline{W}$$

with $w = W \circ Z$, and $P,\ Q$ are C^∞ outside the singular points and bounded near the singular points.
Such equation has nontrivial continuous solutions that are C^∞ outside the singular points.
We can get solution with any regularity by seeking $W(Z) = H(Z)W_1(Z)$ with $H(Z) = \prod_{j=1}^{l}(Z - Z_j)^M$. The function W_1 solves

$$\frac{\partial W_1}{\partial \overline{Z}} = \frac{P(Z)}{\prod_{j=1}^{l}(Z - Z_j)} W_1 + \frac{Q(Z)}{\prod_{j=1}^{l}(Z - Z_j)} \frac{\overline{H}}{H} \overline{W}$$

For given $k \in \mathbb{Z}^+$, we can make W of class C^k at the Z_j's by taking M large enough.

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Theorem ([A.M.] (2013))

Let S be as before. Then for every $\epsilon > 0$ and for every $k \in \mathbb{Z}^+$, there exist surfaces Σ^+ and Σ^- of class C^k in the ϵ-neighborhood of S (for the C^k topology)such that Σ^+ and Σ^- are isometric but not congruent.

Proof. Let Σ_σ and $\Sigma_{-\sigma}$ be surfaces defined by

$$R_{\pm\sigma}(s,t) = R(s,t) \pm \sigma U(s,t)$$

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Let S be as before. Then for every $\epsilon > 0$ and for every $k \in \mathbb{Z}^+$, there exist surfaces Σ^+ and Σ^- of class C^k in the ϵ-neighborhood of S (for the C^k topology) such that Σ^+ and Σ^- are isometric but not congruent.

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Original Riemann Problem: Find a holomorphic function $h = u + iv$ in a domain $\Omega \subset \mathbb{C}$ such that

$$\alpha(t)u(t) + \beta(t)v(t) = \gamma(t) \quad t \in \partial\Omega$$

where α, β, γ are given continuous $\mathbb{R}$-valued function on $\partial\Omega$.

Hilbert reduced it into finding holomorphic functions Φ^+ and Φ^- in interior and exterior of Ω such that

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Can be solved in closed form using Cauchy type integrals

$$\Phi(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\phi(t)}{z-t}\,dt$$

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Many problems can be reduced into solving RH problems.

Elasticity; Airfold theory; Helmholtz Equation; Radon Transform; Inverse Scattering; ...
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RH problem for vector fields

L complex vector field in $\mathbb{R}^2$ Ω is simply connected with smooth boundary

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Schwarz formula: If $h(z) \in C(\overline{\mathbb{D}})$ is holomorphic in the unit disc $\mathbb{D}$ then

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For the RH problem

$$Lw = F(x, y, w) \quad \text{in } \Omega \quad \text{and } \mathrm{Re}(w) = \phi \quad \text{on } \partial\Omega$$

with

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