III. Obstructions to Solvability

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Recall: Classical problem

Problem. Given $\lambda : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$, Hölder continuous, find $w \in C(\overline{\mathbb{D}})$ holomorphic in $\mathbb{D}$ such that $\mathrm{Re}(\overline{\lambda}\, w) = 0$ on $\partial\mathbb{D}$.

The problem has nontrivial solutions only when the index κ of λ is ≥ 0.

$$w(z) = z^{\kappa} e^{i\gamma(z)} \left(i d_0 + \sum_{j=1}^{\kappa} d_j z^j - \overline{d_j}\, \frac{1}{z^j} \right)$$

with $d_0 \in \mathbb{R}$, $d_j \in \mathbb{C}$, and $\gamma(z) = \mathcal{S}(\arg(z^{-\kappa}\lambda(z)))$

$$\mathcal{S}(\psi)(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \psi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}$$

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$L = A\dfrac{\partial}{\partial x} + B\dfrac{\partial}{\partial y}$, locally solvable, nonsingular, C^ω, in $\widetilde{\Omega}$, open subset of $\mathbb{R}^2$.

Allow L to have one dimensional orbits.

$\Omega \subset\subset \widetilde{\Omega}$, Ω simply connected.

$\Lambda : \partial\Omega \longrightarrow \partial\mathbb{D}$, Hölder continuous.

Problem 1. $Lu = 0$ in Ω, and $\mathrm{Re}\left(\overline{\Lambda}\, u\right) = 0$ on $\partial\Omega$.

Problem 2. $Lu = 0$ in Ω, and $\mathrm{Re}\left(\overline{\Lambda}\, u\right) = \phi$ on $\partial\Omega$.with $\phi \in C^\alpha(\partial\Omega, \mathbb{R})$.

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For simplicity, assume that L has no closed one dimensional orbits.

Each one dimensional orbit is a curve Γ_j joining two distinct points on $\partial\Omega$.

$$\Omega \setminus \bigcup_{j=1}^{N} \Gamma_j = \Omega_1 \cup \cdots \cup \Omega_{N+1}$$

$\partial\Omega_j = A_{j1} \cup \Gamma_{j1} \cup A_{j2} \cup \Gamma_{j2} \cup \cdots \cup A_{jm_j} \cup \Gamma_{jm_j}$

A_{jl} arc of $\partial\Omega$; Γ_{jl} one dimensional orbit.

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Associate indices to $\Lambda \in C^\alpha(\partial\Omega, \partial\mathbb{D})$

To each connected component Ω_j, associate and $\kappa_j \in \mathbb{Z}$.
Jump of Λ along orbit $\Gamma_{jk} = \mathrm{arc}(p_{jk}^-, p_{jk}^+)$:
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$$q_{jk} = \left[\frac{\vartheta_{jk}}{\pi}\right] \in \mathbb{Z}, \quad \alpha_{jk} = \frac{\vartheta_{jk}}{\pi} - q_{jk} \in [0,\ 1)\,.$$

$n_j = \#$ of orbits on $\partial\Omega_j$ with q_{jk} odd.
Define

$$\kappa_j = \mathrm{Ind}\left(\Lambda, \partial\Omega_j\right) = \frac{1}{2}\left(\sum_{k=1}^{m_j} q_{jk} - n_j\right)$$

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Theorem (A. Ainouz, K. Boutarene, A.M (2014))

Problem $Lu = 0$ in Ω, $Re(\overline{\Lambda}\, u) = 0$ on $\partial\Omega$ has nontrivial solutions if and only if there is j with $2\kappa_j \geq \delta_j$. Furthermore, the number of independent solutions is

$$r = \sum_{2\kappa_j \geq \delta_j} (2\kappa_j - \delta_j + 1)$$

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Assume $\delta_j = 0$ for all j. Problem $Lu = 0$ in Ω, $Re(\overline{\Lambda}\, u) = \phi$ on $\partial\Omega$ has solution $u \in C^{\mu}(\overline{\Omega})$ except (possibly) at isolated points. The singular points of u consist of a single pole of order $-\kappa_j$ in each component Ω_j with $\kappa_j < 0$. Moreover, the value u on the orbit $\Gamma = arc(p^-, p^+)$ is uniquely determined:

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$$L = (1 + 2ixy)\frac{\partial}{\partial y} - i(x^2 - 1)\frac{\partial}{\partial x}.$$

$x = 1$ and $x = -1$ are one dimensional orbits of L.
First integral $Z(x, y) = x + iy(x^2 - 1)$.
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$\Omega_1 = \Omega \cap \{x > 1\}$, $\Omega_2 = \Omega \cap \{-1 < x < 1\}$, $\Omega_3 = \Omega \cap \{x < -1\}$.

Let $\Lambda(2\mathrm{e}^{i\theta}) = \mathrm{e}^{if(\theta)}$, with

$$f : [0,\ 2\pi] \longrightarrow \mathbb{R}, \quad f(2\pi) = f(0) + \pi q_0, \ \ q_0 \in 2\mathbb{Z}.$$

Jumps: $\vartheta_1 = f(\pi/3) - f(5\pi/3)$, $\vartheta_2 = f(4\pi/3) - f(2\pi/3)$

$$q_1 = \left[\frac{\vartheta_1}{\pi}\right], \ \ q_2 = \left[\frac{\vartheta_2}{\pi}\right]$$

$n_1 = 1$ if q_1 is odd and $n_1 = 0$ if not; $n_2 = 2$ if both q_1, q_2 are odd, $n_2 = 1$ if only one odd, and $n_2 = 0$ if both even;$n_3 = 1$ if q_2 odd and $n_3 = 0$ if not.

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He conjectured that the triviality of $\mathrm{H}^p_\omega(\mathcal{E}^{n+1})$ is equivalent the triviality of $(p-1)$ homology group of the generic fiber of $x + i\phi(x,t)$.

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In the mid 70's Treves (Annal Math) initiated the problem of characterizing $\phi(x,t)$ for which $\mathrm{H}^p_\omega(\mathcal{E}^{n+1})$ is trivial. That is

Equation $\mathbb{L}u = f$ *can be solved with* $u \in \Lambda^{p-1,0}(\mathcal{E}^{n+1})$ *for every* $f \in \Lambda^{p,0}(\mathcal{E}^{n+1})$ *satisfying* $\mathbb{L}f = 0$

He conjectured that the triviality of $\mathrm{H}^p_\omega(\mathcal{E}^{n+1})$ is equivalent the triviality of $(p-1)$ homology group of the generic fiber of $x + i\phi(x,t)$.

The completed proof of the conjecture was given by Cordaro and Hounie in 2001.

Some articles that were written about the problem include

Cordaro-Treves (1991): " $\Longrightarrow$ " for all p

Mendoza-Treves (1991): " $\Longleftarrow$ " for $p = 1$

Chanillo-Treves (1997): " $\Longleftarrow$ " for all p when ω is real analytic.

Cordaro-Hounie (1999): " $\Longleftarrow$ " for $p = n$

$\mathrm{H}^p_\omega(\mathcal{E}^{n+1})$ computed in [A.M.] Trans. AMS in 1997 when ω is of Mizohata type:

$$\omega = d(x + iq(t)), \quad q(t) = t_1^2 + \cdots + t_k^2 - t_{k+1}^2 - \cdots t_n^2$$

Ideas in [A.M] 1997 can used for $\omega = d(x + i\phi(t))$ where $\phi \in \mathcal{E}^n$ has an algebraically isolated singularity at $0 \in \mathbb{R}^n$.
Let $L = |\phi|^{-1}(c)$ with $c > 0$ small. Then

Theorem

$H^p_\omega(\mathcal{E}^{n+1}) \cong H^{p-1}(L) \otimes \mathcal{S}$

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$H^p_\omega(\mathcal{E}^{n+1}) \;\cong\; H^{p-1}(L) \otimes \mathcal{S}$

Ideas in the proof

- Reduction to flatness along the separatrix $\mathbb{R} \times V$ with $V = \phi^{-1}(0)$:

$$\mathrm{H}^p_\omega(\mathcal{E}^{n+1}) \cong \mathrm{H}^p_\omega(\mathcal{F}_0(\mathbb{R} \times V))$$

- Reduction to wedge

$$\mathrm{H}^p_\omega(\mathcal{F}_0(\mathbb{R} \times V)) \cong \oplus_{i=1}^N \mathrm{H}^p_\omega(C_0(W_i))$$

  where $W_1, \cdots, W_N$ are components of $\mathbb{R}^{n+1} \backslash \mathbb{R} \times V$ and $C_0(W_i)$ is the space of germs at 0 of $C^\infty(\overline{W_i})$ that are flat along $\overline{W_i} \cap \mathbb{R} \times V$.
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Space $\mathcal{S}$

Two germs f_1, f_2 of C^∞ at $0 \in \overline{\mathbb{R}^2_+}$ $(y \geq 0)$ are equivalent if

$$\int_{\mathbb{C}} \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - x}\, d\zeta \wedge d\bar{\zeta} \quad \text{is real analytic function } (x \in \mathbb{R})$$

$\mathcal{S}$: space of equivalence classes, $\mathcal{S} = C^\infty(\overline{\mathbb{R}^2_+}, 0)/\sim$.

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Proof

To $\eta = fd\overline{z} + gdz \in \Lambda^1(C^\infty(\overline{\mathbb{R}^2_+})$ associate $[f] \in \mathcal{S}$.
Since the general solution of $u_{\overline{z}} = f$ in $y \geq 0$ is $u(z) = u_0(z) - h(z)$, with h holomorphic in $y > 0$ and smooth up to $y \geq 0$ and

$$u_0(z) = \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} \, d\zeta \wedge d\overline{\zeta}$$

then in order for u to be flat along $y = 0$ we need $u_0(x) = h(x)$.

$$\Phi : \ \mathrm{H}^1_{d(x+iy)}(\mathcal{F}_0(\{y=0\})) \longrightarrow \mathcal{S}\,, \quad \Phi([\eta]) = [f]$$

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Cohomology with formal coefficients

$F(x, t, s) = x + i(\phi(t) + s)$.

$\mathcal{E}_x$: germs at $0 \in \mathbb{R}$ of C^∞ functions

$\mathcal{E}_x[t, s]$: ring of formal power series in t, s with coefficients in $\mathcal{E}_x$.

$$\Lambda^\bullet_{dF}(\mathcal{E}_x[t, s]) = \frac{\Lambda^\bullet(\mathcal{E}_x[t, s])}{\Lambda^{\bullet-1}(\mathcal{E}_x[t, s]) \wedge dF}$$

Proposition

$H^0_{dF}(\mathcal{E}_x[t, s]) \cong \mathcal{E}_x$ and $H^p_{dF}(\mathcal{E}_x[t, s]) \cong 0$ for $p > 0$.

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Reduction to flatness along x-axis

Proposition

$\omega = d(x + i\phi(t))$. If $\eta \in Z^p_\omega(\mathcal{E}^{n+1})$, then there exists $u \in \Lambda^{p-1}(\mathcal{E}^{n+1})$ such that

$$\eta - du \in Z^p_\omega(\mathcal{F}_0(\{t = 0\}))$$

Proof

$d\eta \wedge \omega = 0 \implies d\eta = \eta_1 \wedge \omega$ for some $\eta_1 \in \Lambda^p(\mathcal{E}^{n+1})$
$d^2\eta = d\eta_1 \wedge \omega = 0 \implies d\eta_1 = \eta_2 \wedge \omega$ for some $\eta_2 \in \Lambda^p(\mathcal{E}^{n+1})$
The continuation gives rise to a (Godbillion-Vey) sequence $\{\eta_j\}_j \subset \Lambda^p(\mathcal{E}^{n+1})$ with $d\eta_j = \eta_{j+1} \wedge \omega$. (set $\eta = \eta_0$).
Let α be the formal p-form in (x, t, s):

$$\alpha = \sum_{j \geq 0} \eta_j \frac{(is)^j}{j!}$$

so that $d\alpha = \left(\sum \eta_{j+1} \frac{(is)^j}{j!}\right) \wedge dF$.

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$d\eta \wedge \omega = 0 \implies d\eta = \eta_1 \wedge \omega$ for some $\eta_1 \in \Lambda^p(\mathcal{E}^{n+1})$
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Let α be the formal p-form in (x, t, s):

$$\alpha = \sum_{j \geq 0} \eta_j \frac{(is)^j}{j!}$$

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so (from previous proposition) $\exists \widehat{\beta} \in \Lambda^{p-1}(\mathcal{E}_x[t,s])$ such that $(T_{t,s}\alpha - d\widehat{\beta}) \wedge dF = 0$.
Let $\beta \in \Lambda^{p-1}(\mathcal{E}^{n+2})$ s.t. $T_{t,s}\beta = \widehat{\beta}$ (Borel).
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$\phi(t) \in \mathcal{E}^n$ is $\mathbb{R}$-valued with an algebraically isolated singularity at 0.

$S_\epsilon^{n-1} = \{x \in \mathbb{R}^n : \ |x| = \epsilon\}$
$B_\epsilon^n = \{x \in \mathbb{R}^n : \ |x| < \epsilon\}.$
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For $\epsilon > 0$ small enough, there exists $\delta > 0$ s.t.

- S_ϵ^{n-1} *is transversal to hypersurfaces* $\phi^{-1}(t)$ *for* $|t| < \delta$.
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Complex on product manifold

M: smooth manifold of dimension m
$R^+(\epsilon, \delta) = \{(x, y) \in \mathbb{R}^2 : \ |x| < \epsilon, \ 0 \leq y < \delta\}$
$Y = \{(p, x, y) \in M \times R^+(\epsilon, \delta) : \ y = 0\}$
$\mathcal{F}(Y)$: germs of flat functions along Y in $C^\infty(M \times \mathbb{R}^+(\epsilon, \delta))$.

$$\Lambda^\bullet_{d(x+iy)}(\mathcal{F}(Y)) = \frac{\Lambda^\bullet(\mathcal{F}(Y))}{\Lambda^{\bullet-1}(\mathcal{F}(Y)) \wedge d(x+iy)}$$

Proposition

$H^p_{dz}(\mathcal{F}(Y)) \cong H^{p-1}(M) \times \mathcal{S}$

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$\mathcal{F}(Y)$: germs of flat functions along Y in $C^\infty(M \times \mathbb{R}^+(\epsilon,\delta))$.

$$\Lambda^\bullet_{d(x+iy)}(\mathcal{F}(Y)) = \frac{\Lambda^\bullet(\mathcal{F}(Y))}{\Lambda^{\bullet-1}(\mathcal{F}(Y)) \wedge d(x+iy)}$$

Proposition

$H^p_{dz}(\mathcal{F}(Y)) \cong H^{p-1}(M) \times \mathcal{S}$

Complex on product manifold

M: smooth manifold of dimension m
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Proof

$\eta = \eta^0 + \eta^1 \wedge d\bar{z} \mod(dz)$: p-form on $M \times \mathbb{R}^+(\epsilon, \delta)$ with η^0 and η^1, p and $(p-1)$ forms on M depending on parameter z.

$d\eta = d_0\eta^0 + (d_0\eta^1 + (-1)^p \eta^0_{\bar{z}}) \wedge d\bar{z} \mod(dz)$

If η is dz-closed, then $d_0\eta^0 = 0$ and $d_0\eta^1 + (-1)^p\eta^0_{\bar{z}} = 0$.

$\sigma_1, \cdots, \sigma_l \in \Lambda^p(C^\infty(M))$: generators of $H^p(M)$

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Reduction to wedges

Let $W = (-a,\ a) \times U_{\epsilon,\delta} \subset \mathbb{R} \times \mathbb{R}^n$
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For $p > 1,\ \ H^p_\omega(\mathcal{E}^{n+1}) \cong \oplus_{i=1}^N H^p_\omega(C_0^\infty(W_i))$

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$W \backslash \mathbb{R} \times V = W_1 \cup \cdots \cup W_N$.
$L = |\phi|^{-1}(\delta)$, L_i connected component of L such that $(-a,\ a) \times L_i \subset W_i$.
$C_0^\infty(W_i)$: space of germs at 0 of C^∞ functions with support $\subset \overline{W_i}$.

Proposition

For $p > 1$, $H^p_\omega(\mathcal{E}^{n+1}) \cong \oplus_{i=1}^N H^p_\omega(C_0^\infty(W_i))$

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Proposition

For $p > 1$, $H_\omega^p(\mathcal{E}^{n+1}) \cong \oplus_{i=1}^N H_\omega^p(C_0^\infty(W_i))$

Proof

First if $\eta \in Z^p_\omega(\mathcal{E}^{n+1})$, we can prove that
$\eta = du + \eta^0 \mod(\omega)$ with $\eta^0 \in Z^p_\omega(\mathcal{F}_0(\mathbb{R} \times V)$
Define $\eta_i^0 = \begin{cases} \eta^0 & \text{in } W_i \\ 0 & \text{in } \mathbb{R}^{n+1} \backslash W_i \end{cases}$
so $\eta^0 = \sum_i \eta_i^0$.

Second we prove that $\eta_i^0 \in \mathrm{B}^p_\omega(\mathcal{E}^{n+1}) \Longrightarrow \eta_i^0 \in \mathrm{B}^p_\omega(C_0^\infty(W_i))$
The map $[\eta] \longrightarrow \sum_i [\eta_i^0]$ gives an isomorphism

$$H^p_\omega(\mathcal{E}^{n+1}) \longrightarrow \bigoplus_i H^p_\omega(C_0^\infty(W_i))$$

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$H^p_\omega(C_0^\infty(W_i)) \cong H^{p-1}(L_i) \times \mathcal{S}$

Proof. Roche's Theorem induces an isomorphism

$$\Psi^* : \mathcal{F}((-a, a) \times L_i \times 0)\Lambda^\bullet(C^\infty((-a, a) \times L_i \times [0, \delta)) \longrightarrow \Lambda^\bullet(C_0^\infty(W_i))$$

which in turn induces an isomorphism

$$\mathrm{H}^{p-1}(L_i)\times\mathcal{S} \cong \mathrm{H}^p_{d(x+iy)}(\mathcal{F}(L_i\times R^+(\epsilon,\delta))) \cong \mathrm{H}^p_{d(x+i\phi(t))}(C_0^\infty(W_i))$$

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