IV. Small Divisors and Solvability on Tori

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Liouville numbers

$\lambda \in \mathbb{R} \backslash \mathbb{Q}$ is a Liouville number if

$\forall N \in \mathbb{N},\ \exists C > 0$ and a sequence $\{(p_j, q_j)\}_j \subset \mathbb{Z} \times \mathbb{N}$ such that $q_j \longrightarrow \infty$ and

$$\left|\lambda - \frac{p_j}{q_j}\right| < \frac{C}{q_j^N} \quad \forall j \in \mathbb{N}$$

Examples. Let $b \in \mathbb{N}$. Then $\lambda = \sum_{n=1}^{\infty} \frac{1}{b^{n!}}$ is Liouville.

An algebraic number is not Liouville.

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$$\mathbb{T}^2 = \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)^2 .$$

$u \in \mathcal{D}'(\mathbb{T}^2)$. $u = \sum_{j,k} \widehat{u}(j,k)\mathrm{e}^{i(jx+kt)}$.

There exist constants $C, M > 0$ such that

$$|\widehat{u}(j,k)| \leq C(1+j^2+k^2)^M .$$

$$u \in C^\infty(\mathbb{T}^2) \iff \sup \frac{|\widehat{u}(j,k)|}{(1+j^2+k^2)N} < \infty \quad \forall N \in \mathbb{N}$$

$P = \sum_{p,q=0}^{n} c_{p,q} \frac{\partial^{p+q}}{\partial x^p \partial t^q}$ with $c_{p,q} \in \mathbb{C}$. P is Globally Hypoelliptic (GH) if

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Greenfield-Wallach Theorem

Theorem (S. Greenfield and N. Wallach (1972))

P is (GH) if and only if there exist $A,\ B > 0$ such that

$$\left|\sum_{p,q=0}^{n} c_{p,q} j^p k^q\right| \geq \frac{A}{(j^2+k^2)^B} \quad \textit{for } |j| \textit{ and } |k| \textit{ large}$$

In case of a vector field $L = \dfrac{\partial}{\partial t} - \lambda \dfrac{\partial}{\partial x}$ with $\lambda \in \mathbb{R}\backslash\mathbb{Q}$

Theorem (S. Greenfield and N. Wallach (1972))

L is (GH) if and only if λ is not Liouville.

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Impact of Greenfield-Wallach Theorem

Some papers

- Hounie (1979): A two-dimensional surface with a (GH) L vector field is a torus and $L \sim \partial_t - \lambda \partial_x$.
- Chen-Chi (2000): On $\mathbb{T}^n$ a (GH) vector field is $\sim \sum_j A_j \partial_{x_j}$ with the A_j's in $\mathbb{R}$ satisfying a Diophantine condition.
- Bergamasco-Cordaro-Malagutti (1993): (GH) on $M \times \mathbb{S}^1$ with M compact
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Diophantine condition for matrices

$M = (a_{kj}) \in \mathcal{M}_{\mathbb{R}}(m,n)$

$N_M = M(\mathbb{Z}^n) \cap \mathbb{Z}^m$. $N_M \leq M(\mathbb{Z}^n)$ with rank $r \leq \min(m,n)$.

$J_1, \cdots, J_r \in \mathbb{Z}^n$ such that $\mathrm{span}\{MJ_1, \cdots, MJ_r\} = N_M = M(N_0)$

$N_0 = \mathrm{span}\{J_1, \cdots, J_r\} \leq \mathbb{Z}^n$.

(DC) there exist $C > 0$ and $\rho > 0$ such that

$$\max_{1 \leq \ell \leq m} \{|k_\ell + a_\ell \cdot J|\} \geq C(|K| + |J|)^{-\rho},$$

for all $(K, J) \in \mathbb{Z}^m \times \mathbb{Z}^n$ such that $K + MJ \neq 0$, where $a_\ell = (a_{\ell 1}, \cdots, a_{\ell n})$ is the ℓ-th row of M and $K = (k_1, \cdots, k_m)$.

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Diophantine condition for matrices

$M = (a_{kj}) \in \mathcal{M}_{\mathbb{R}}(m, n)$
$N_M = M(\mathbb{Z}^n) \cap \mathbb{Z}^m$. $N_M \leq M(\mathbb{Z}^n)$ with rank $r \leq \min(m, n)$.
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Examples

- $M \in \mathcal{M}_{\mathbb{Q}}(m,n)$ or $\in \mathcal{M}_{\mathbb{A}}(m,n)$ satisfy (DC).
- $m = n = 1$, $M = \lambda$ satisfies (DC) if and only if λ is not a Liouville number
- Let $\lambda \in \mathbb{R}$ be a Liouville number $X \in \mathbb{Z}^m$, $Y \in \mathbb{Z}^n$ with $Y_1 = 1$, and $L \in \mathcal{M}_{\mathbb{Z}}(m,n)$.
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Let $M \in \mathcal{M}_{\mathbb{R}}(m,n)$ and $S \in SL(m,\mathbb{Z})$.

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A differential complex

$\Omega = \{\omega^1, \cdots, \omega^n\} \subset C^\infty(\mathbb{T}^m, \Lambda^1)$ with $d\omega^j = 0$.

$C^\infty(\mathbb{T}^{m+n}, \Lambda^{p,0})$: space of p-forms f on torus $\mathbb{T}^{m+n} = \mathbb{T}^m \times \mathbb{T}^n$ with

$$f = \sum_{|L|=p} f_L(t,x)\, dt_K \quad f_L \in C^\infty(\mathbb{T}^{m+n}, \mathbb{C})$$

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$$\mathbb{L}_\Omega^2 = 0$$

$Z_\Omega^p(\mathbb{T}^{m+n})$: space of Ω-closed p-forms ($\mathbb{L}_\Omega f = 0$)
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Matrix of periods

$\gamma_1, \cdots, \gamma_m$ closed loops in $\mathbb{T}^m$ such that the homotopy classes $\Gamma = \{[\gamma_1], \cdots, [\gamma_m]\}$ is a basis of $\mathrm{H}_1(\mathbb{T}^m)$.

To $\Omega = \{\omega_1, \cdots, \omega_n\}$, associate the $m \times n$ matrix

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Theorem ([P. Dattori, A.M.])

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$$\mathbb{L}_{\Omega_0}S_Hf = \mathbb{L}_{\Omega_0}g = \sum_J \left(d_t\widehat{g}(t,J) + i\sum_k j_k\omega_0^k\widehat{g}(t,J)\right) \mathrm{e}^{iJ\cdot x}$$

$$d_t\widehat{g}(t,J) = \mathrm{e}^{iJ\cdot H(t)} \left[d_t\widehat{f}(t,J) + i\sum_k j_k dh_k(t)\wedge\widehat{f}(t,J)\right]$$

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$$S_{-H}\mathbb{L}_{\Omega_0}S_Hf = \sum_J \mathbb{L}_{\Omega}\left(\widehat{f}(t,J)\mathrm{e}^{iJ\cdot x}\right) = \mathbb{L}_{\Omega}f.$$

Consequence

$$S_HZ^*_{\Omega}(\mathbb{T}^{m+n}) = Z^*_{\Omega_0}(\mathbb{T}^{m+n}) \quad \text{and} \quad S_HB^*_{\Omega}(\mathbb{T}^{m+n}) = B^*_{\Omega_0}(\mathbb{T}^{m+n})$$

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$f \in C^\infty(\mathbb{T}^{m+n}, \Lambda^{p,0})$,

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Proposition

Suppose that M satisfies condition (DC). Then f is Ω_0-exact if and only if for every $J \in N$, $e^{iMJ \cdot t}\widehat{f}(t, J)$ is an exact form in $\mathbb{T}^m$

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Proof

"$\Longrightarrow$" Suppose $\mathbb{L}_{\Omega_0} u = f$. Use partial Fourier series wrt x.

$$d\widehat{u}(t,J) + i\sum_{s=1}^{m} j_s w_0^s \wedge \widehat{u}(t,J) = \widehat{f}(t,J)$$

For $J \in N$, $e^{iMJ\cdot t} \in C^\infty(\mathbb{T}^m)$ and
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$\mathrm{e}^{iMJ\cdot t}\widehat{f}(t,J)$ exact $\implies f(-MJ,J) = 0$

We can define u_N as

$u_N = \sum \widehat{u}_N(K,J)\mathrm{e}^{iK\cdot t}\mathrm{e}^{iJ\cdot x}$ with

$$\widehat{u}_N(K,J) = \sum_{|L|=p,\ \sigma\in L} \frac{\epsilon}{|K+MJ|}\widehat{f}(K,J)\, dt_{L\setminus\sigma} \text{ with } \epsilon = \pm 1$$

Condition (DC) and $\widehat{f}(K,J)$ with fast decay $\implies u_N \in C^\infty$

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Proof of $H^p_{\Omega_0}(\mathbb{T}^{m+n}) \cong H^p(\mathbb{T}^m) \otimes \Psi^*(C^\infty(\mathbb{T}^r))$

$f \in Z^p_{\Omega_0}(\mathbb{T}^{m+n})$, $d(\mathrm{e}^{iMJ\cdot t}\widehat{f}(t,J) = 0 \quad \forall J \in N.$

$$\mathrm{e}^{iMJ\cdot t}\widehat{f}(t,J) = \sum_{|L|=p} C^L_J \, dt_L + \text{ exact form}$$

$C^L_J \in \mathbb{C}$ with rapid decay.
Let $H : \mathbb{Z}^r \longrightarrow N \leq \mathbb{Z}^n$ isomorphism. For every L let

$$g_L(s) = \sum_{\nu\in\mathbb{Z}^r} C^L_{H\nu}\,\mathrm{e}^{i\nu s} \in C^\infty(\mathbb{T}^r)$$

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Semiglobal solvability

Solvability of the equation

$$Lu = f$$

L is a vector field in a neighborhood of an orbit

- A necessary and sufficient condition for local solvability is the Nirenberg-Treves Condition (P)
- Condition (P) is also sufficient for solvability in a neighborhood of a nonrelatively compact orbit of L (L. Hörmander, Ann. Math. 1978, J. Hounie, Proc. AMS, 1985)
- Solvability in a neighborhood of a relatively compact orbit not yet well understood.
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A class of vector fields

Let $\omega_1, \cdots, \omega_m \in \mathbb{R}$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$,
$t = (t_1, \cdots, t_m) \in \mathbb{T}^m$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

$$\begin{aligned} X &= \sum_{k=1}^{m} (\omega_k + g_k(t,x)) \frac{\partial}{\partial t_k} \\ Y &= \sum_{j=1}^{n} (\lambda_j x_j + f_j(t,x)) \frac{\partial}{\partial x_j}, \end{aligned}$$

where $g_k, f_j \in C^a(\mathbb{T}^m \times B(r); \mathbb{R})$, with $a = \infty$ (smooth case) or $a = \varpi$ (real analytic case) and such that

$$g_k = O(|x|) \quad \text{and} \quad f_j = O(|x|^2)$$

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Notation

$$X = (\omega + g(t,x))\partial_t \qquad Y = (\Lambda x + f(t,x))\partial_x$$

where

$$g = (g_1, \cdots, g_m) \quad f = (f_1, \cdots, f_n),$$
$$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$
$$\partial_t = \left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m}\right)^T \quad \partial_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T$$

Let $\mathbb{T}_0^m = \mathbb{T}^m \times \{0\} \subset \mathbb{T}^m \times \mathbb{R}^n$

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$$g = (g_1, \cdots, g_m) \quad f = (f_1, \cdots, f_n),$$
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$$\partial_t = \left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m}\right)^T \quad \partial_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)^T$$

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Note that since $Y = 0$ and $X = \omega\partial_t$ on $\mathbb{T}_0^m$, then for every $(t, 0) \in \mathbb{T}_0^m$ the orbit of X through $(t, 0)$ is also an orbit of the real vector $X + Y$ and of the complex vector field $X + iY$.
These orbits are trapped on the torus $\mathbb{T}_0^m$.
Our objective is to understand the equations

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Siegel type condition $(\mathcal{S}_{\mathbb{R}})$

The pair $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfies the Siegel type condition $(\mathcal{S}_{\mathbb{R}})$ if the following holds:

$(\mathcal{S}_{\mathbb{R}})$: There exists $C > 0$ and $\mu > 0$ such that

$$|i < \omega, K > + < \lambda, J > -\epsilon\lambda_j| \geq \frac{C}{(|K| + |J|)^{\mu}}$$

for every $K \in \mathbb{Z}^m$, $J \in \mathbb{N}^n$, $j = 1, \cdots, n$, $\epsilon = 0$ or 1 and $|K| + |J| > 0$.

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$$< \omega, K >= \omega_1 k_1 + \cdots + \omega_m k_m \,, \quad < \lambda, J >= \lambda_1 j_1 + \cdots + \lambda_n j_n \,.$$

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Normalization of $X + Y$

Theorem

Suppose that (ω, λ) satisfies $(\mathcal{S}_{\mathbb{R}})$. Then there exists a diffeomorphism Φ of class C^a

$$\Phi(t, x) = (t + \phi(t, x), x + \psi(t, x))$$

defined in a neighborhood of $\mathbb{T}_0^m$, with $\phi(t, 0) = 0$, $\psi(t, 0) = 0$, such that

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Idea of the proof: (C^∞-case)

Step 1. Formal equivalence

Step 2. Reduction to normalization up to flat terms

Step 3. Removal of flat terms

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Solvability of $M_{\omega,\lambda}$

$$M_{\omega,\lambda} = \omega\partial_t + \Lambda x \partial_x$$

Theorem

Assume that $(\omega,\lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ satisfy $(\mathcal{S}_{\mathbb{R}})$. Let f be a C^a-function ($a = \varpi$ or $a = \infty$) in a neighborhood of $\mathbb{T}_0^m$ such that

$$\int_{\mathbb{T}^m} f(t,0)dt = 0\,.$$

Then equation $M_{\omega,\lambda}u = f$ has a C^a-solution in a neighborhood of $\mathbb{T}_0^m$

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We seek a diffeomorphism that transforms

$$X + iY = (\omega + g)\partial_t + i(\Lambda x + f)\partial_x$$

into

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Remark. Note that since $\omega\partial_t$ and $\Lambda x\partial_x$ commute, then a necessary condition for such an equivalence to exist is that X and Y need to commute.

$$[X, Y] = 0$$

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Normalization of $X + iY$: analytic case

Theorem

Suppose that

- (ω, λ) *satisfy* $(\mathcal{S}_{\mathbb{C}})$,
- *f and g are real analytic, and*
- $[X, Y] = 0$.

Then there exists a real analytic diffeomorphism

$$\Phi(t, x) = (t + \phi(t, x), x + \psi(t, x))$$

near $\mathbb{T}^m_0$ *with* $\phi(t, 0) = 0$, $\psi(t, 0) = 0$, *such that*

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Normalization of $X + iY$: C^∞-case

Theorem

Suppose that

- (ω, λ) *satisfy* $(\mathcal{S}_{\mathbb{C}})$,
- $\lambda_1, \cdots, \lambda_n$ *have the same sign,*
- f *and* g *are* C^∞, *and*
- $[X, Y] = 0$.

Then there exists a C^∞ *diffeomorphism*

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A non normalizable vector field

Let $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ with $n \geq 2$ such that $\lambda_1 < 0$ and $\lambda_2 > 0$.
Let $\beta(x) \in C^0(\mathbb{R}^n, \mathbb{R})$ defined by

$$\beta(x) = \begin{cases} x_1^{\lambda_2} x_2^{-\lambda_1} & \text{if } x_1 > 0,\ x_2 > 0\,, \\ 0 & \text{elsewhere} \end{cases}$$

Then $\Lambda x \partial_x \beta(x) = 0$.
Let $\alpha(y) \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\alpha(y) > 0$ for $y > 0$ and $\alpha(y) = 0$ for $y \leq 0$.
Define $g(t, x) \in C^\infty(\mathbb{T}^m \times \mathbb{R}^n, \mathbb{R}^m)$ by

$$g_1(t, x) = \alpha(\beta(x)), \quad g_k(t, x) = 0 \ \text{ for } k = 2, \cdots, m.$$

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Consider the vector field

$$T = (\omega + g(t,x))\partial_t + i\Lambda x \partial_x = L_{\omega,\lambda} + \alpha(\beta(x))\frac{\partial}{\partial t_1}$$

Note that $[\mathrm{Re}(T), \mathrm{Im}(T)] = 0$.

Proposition

There is no diffeomorphism Φ near $\mathbb{T}_0^m$ such that

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$$\Phi(t,x) = (t + \phi(t,x), \psi(t,x))$$

such that $\Phi_* T = L_{\omega,\lambda}$. For $j = 1, \cdots, m$, the function $\phi_j(t,x)$ satisfies

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In particular for $j = 1$, the t-periodic function ϕ_1 satisfies

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Solvability of $L_{\omega,\lambda}$: real analytic case

$$L_{\omega,\lambda} = \omega\partial_t + i\Lambda x\partial_x$$

Theorem

Assume that $(\omega, \lambda) \in \mathbb{R}^m \times \mathbb{R}^n$ *satisfy* $(\mathcal{S}_{\mathbb{C}})$. *Let* f *be a* C^{ϖ}*-function in a neighborhood of* $\mathbb{T}^m_0$ *such that*

$$\int_{\mathbb{T}^m} f(t,0)dt = 0\,.$$

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