THE HEAT EQUATION

1. Exercises

In exercises 1 to 5, write the BVP for the temperature $u(x,t)$ in a homogeneous and laterally insulated rod of length L and diffusivity k in the following cases.

Exercise 1. The left end and right end are kept at temperature 0 degrees, the initial temperature at slice x is x degrees, $k = 1$, and $L = 20$.

$$\begin{array}{ll} u_t(x,t) = u_{xx}(x,t) & 0 < x < 20\,, \quad t > 0 \\ u(x,0) = x & 0 < x < 20 \\ u(0,t) = 0, \quad u(20,t) = 0, & t > 0 \end{array}$$

Exercise 2. The left end is kept at temperature 10 degrees, the right end at temperature 50 degrees, the initial temperature at any slice x is 100 degrees, $k = 2$, $L = 50$.

Exercise 3. The left end is insulated, the right end at temperature 50 degrees, the initial temperature at any slice x is x^2 degrees, $k = 1/2$, $L = 50$.

$$\begin{array}{ll} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) & 0 < x < 50\,, \quad t > 0 \\ u(x,0) = x^2 & 0 < x < 50 \\ u_x(0,t) = 0, \quad u(50,t) = 50, & t > 0 \end{array}$$

Exercise 4. Both ends are insulated, the initial temperature at slice x is 100 degrees, $k = 1$, $L = 20$.

Exercise 5. The left end is controlled so that at time t, the temperature is $100\cos t$ degrees, the right end is insulated, the initial temperature is 50 degrees, $k = 1$, $L = 20$.

$$\begin{array}{ll} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) & 0 < x < 20\,, \quad t > 0 \\ u(x,0) = 50 & 0 < x < 20 \\ u(0,t) = 100\cos t, \quad u_x(20,t) = 0, & t > 0 \end{array}$$

Exercise 6. Consider two identical, laterally insulated, uniform rods with diffusivity k and length L. These two rods are joined together to form a new rod of length $2L$ (the right end of rod 1 is joined to the left end of rod 2 as in the figure). Suppose that at time $t = 0$ (just the

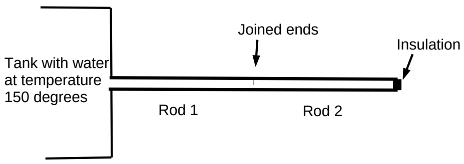

Figure 1. *Two rods joined to form a single rod*

moment when the rods are joined), the temperature of rod 1 is 0 degrees, that of rod 2 is 100

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degrees, the right end of rod 2 in insulated and the left end of rod 1 is immersed in a tank with temperature of 150 degrees. Write the BVP for the temperature of the new rod of length $2L$.

Exercise 7. This time consider three identical, laterally insulated, uniform rods with diffusivity k and length L. These three rods are joined together to form a new rod of length $3L$ (the right end of rod 1 is joined to the left end of rod 2, and the right end of rod 2 is joined to the left end of rod 3 as in the figure). Suppose that the left end of rod 1 is insulated, the right end of

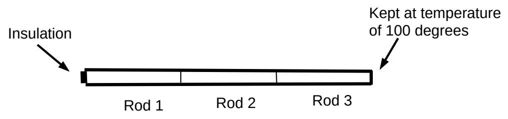

FIGURE 2. *Three rods joined to form a single rod*

rod 3 is kept at temperature of 100 degrees. Suppose that initially, the temperature in rod is 100 degrees, that in rod 2 is given by $100x(1-x/L)$ (here x is the distance from the left end of rod 2 to the cross section), and the temperature in rod 3 is 0 degrees.

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t) & 0 < x < 3L\,, \quad t > 0 \\ u(x,0) = f(x) & 0 < x < 3L \\ u_x(0,t) = 0, \quad u(3L,t) = 100, & t > 0 \end{cases}$$

where

$$f(x) = \begin{cases} 100 & 0 < x < L \\ 100(x-L)(2L-x)/L & L < x < 2L \\ 0 & 2L < x < 3L \end{cases}$$

Exercise 8. The temperature function in a laterally insulated rod made of copper is found (say experimentally) to be $\mathrm{e}^{-0.046t}\cos(0.2x)$. Find the thermal diffusivity k of copper.

Exercise 9. Verify that the function

$$u(x,t) = \mathrm{e}^{-\pi^2 t/50}\sin\frac{\pi x}{10} - 5\mathrm{e}^{-4\pi^2 t/50}\sin\frac{2\pi x}{10}$$

is a solution of the BVP

$$\begin{array}{ll} u_t = 2u_{xx} & 0 < x < 10, \quad t > 0 \\ u(0,t) = u(10,t) = 0 & t > 0 \\ u(x,0) = \sin\dfrac{\pi x}{10} - 5\sin\dfrac{2\pi x}{10} & 0 < x < 10. \end{array}$$

The derivatives u are

$$\begin{aligned} u_t(x,t) &= -\tfrac{\pi^2}{50}\mathrm{e}^{-\pi^2 t/50}\sin\tfrac{\pi x}{10} + 5\tfrac{4\pi^2}{50}\mathrm{e}^{-4\pi^2 t/50}\sin\tfrac{2\pi x}{10} \\ u_{xx}(x,t) &= -\tfrac{\pi^2}{100}\mathrm{e}^{-\pi^2 t/50}\sin\tfrac{\pi x}{10} + 5\tfrac{4\pi^2}{100}\mathrm{e}^{-4\pi^2 t/50}\sin\tfrac{2\pi x}{10} \end{aligned}$$

clearly $u_t = 2u_{xx}$. Since $\sin(k\pi) = 0$ for any $k \in \mathbb{Z}$, then $u(0,t) = u(10,t)$ for $t > 0$. For $t = 0$, we have $u(x,0) = \sin\dfrac{\pi x}{10} - 5\sin\dfrac{2\pi x}{10}$

If there is heat radiation within the rod of length L, then the 1-dimensional heat equation might take the form

$$u_t = ku_{xx} + F(x,t).$$

Exercise 10 to 13 deal with the steady-state situation. This means that the temperature u and F are independent on time t. In particular, $u_t \equiv 0$. The above heat equation becomes just an ordinary differential equation that you have learned how to solve in the first Differential Equation course (MAP3102).

Exercise 10. Find $u(x)$ if $F = 0$ (no radiation), $k = 3$, $u(0) = 2$, $u(L) = 10$.

Exercise 11. Find $u(x)$ if $F = 0$ (no radiation), $k = 1$, $u(0) = 2$, $u'(L) = 2$.

The function $u(x)$ satisfies $u''(x) = 0$, $u(0) = 2$ and $u'(L) = 2$. It follows from $u''(x) = 0$ that $u(x) = Ax + B$ with A, B constants. Then, $u(0) = B = 2$; $u'(x) = A$ and $A = 2$. The solution is $u(x) = 2x + 2$.

Exercise 12. Find $u(x)$ if $F(x) = x$, $k = 1$, $u(0) = 0$, $u(L) = 0$.

Exercise 13. Find $u(x)$ if $F(x) = \sin \dfrac{\pi x}{L}$, $k = 2$, $u(0) = u'(0)$, $u(L) = 1$.

The function $u(x)$ satisfies $u''(x) = -\dfrac{\sin(\pi x/L)}{2}$, $u(0) = u'(0)$ and $u(L) = 1$. A particular solution of the differential equation is $-\dfrac{L^2}{2\pi^2}\sin(\pi x/L)$ and the general solution of the DE is

$$u(x) = Ax + B - \frac{L^2}{2\pi^2}\sin(\pi x/L)$$

To find the constants we use $u(0) = u'(0)$ and $u'(L) = 1$ and find

$$B = A - \frac{L}{2\pi}; \qquad AL + B = 1$$

It follows that $A = \dfrac{2\pi + L}{2\pi(L+1)}$, $B = \dfrac{2\pi - L^2}{2\pi(L+1)}$ and the solution

$$u(x) = \frac{2\pi + L}{2\pi(L+1)}x + \frac{2\pi - L^2}{2\pi(L+1)} - \frac{L^2}{2\pi^2}\sin\frac{\pi x}{L}$$

Exercises 14 to 18, deal with the temperature $u(x, y, t)$ in a homogeneous and thin plate. We assume that the top and bottom of the plate are insulated and the material has diffusivity k. Write the BVP in the following cases.

Exercise 14. The plate is a square with side length 1 with $k = 1$. The horizontal sides are kept at temperature 0 degrees, the vertical sides at temperature 100 degrees. The initial temperature in the plate is 50 degrees (constant throughout the plate).

Exercise 15. The plate is a 1×2 rectangle with $k = 2$. The vertical left side and the top horizontal sides are insulated. The vertical right and the horizontal bottom sides are kept at temperatures 0 and 100 degrees, respectively. The initial temperature distribution in the plate is $f(x, y) = \sin x \cos y$.

$$\begin{cases} u_t(x,y,t) = 2(u_{xx}(x,y,t) + u_{yy}(x,y,t)) & 0 < x < 1, \quad 0 < y < 2, \quad t > 0; \\ u_x(0,y,t) = 0, \quad u(1,y,t) = 0 & 0 < y < 2, t > 0; \\ u(x,0,t) = 100, \quad u_y(x,2,t) = 0 & 0 < x < 1, \quad t > 0; \\ u(x,y,0) = \sin x \cos y & 0 < x < 1, \quad 0 < y < 2. \end{cases}$$

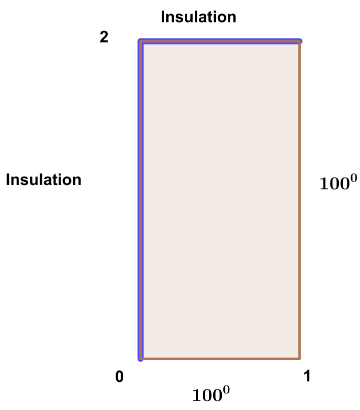

FIGURE 3. *Rectangular plate*

Exercise 16. The plate is triangular as in the figure. The vertical side is kept at temperature

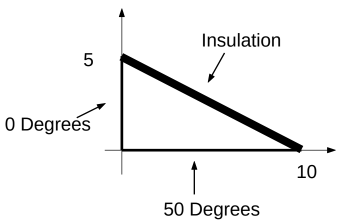

FIGURE 4. *Triangular plate*

0 degrees, the horizontal side at 50 degrees, and the slanted side is insulated. The initial temperature is 100 degrees throughout.

Exercise 17. The plate is triangular as in the figure. The vertical side is kept at temperature

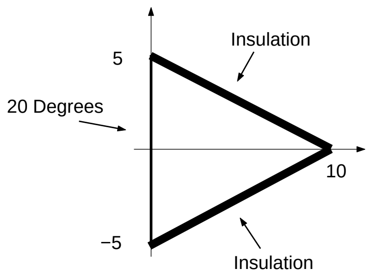

FIGURE 5. *Triangular plate*

20 degrees, the slanted sides are insulated and the initial temperature is given by the function xy.

The region occupied by the plate is given $\{(x, y) \in \mathbb{R}^2 : \ 0 \le x \le 10, \ \ \frac{x}{2} - 5 \le y \le -\frac{x}{2} + 5\}$. The unit normal to the slanted side $y = \frac{x}{2} - 5$ is $\overrightarrow{N_1} = \frac{<1,-2>}{\sqrt{5}}$ and the unit normal to the slanted side $y = -\frac{x}{2} + 5$ is $\overrightarrow{N_2} = \frac{<1,2>}{\sqrt{5}}$. Insulations along the slanted sides are given by

$$\overrightarrow{\text{grad}u} \cdot \overrightarrow{N_1} = 0 \iff u_x - 2u_y = 0$$
$$\overrightarrow{\text{grad}u} \cdot \overrightarrow{N_2} = 0 \iff u_x + 2u_y = 0$$

The BVP satisfied by the temperature function $u(x, y, t)$ is:

$$\begin{cases} u_t = k(u_{xx} + u_{yy}) & 0 < x < 10, \quad \frac{x}{2} - 5 \le y \le -\frac{x}{2} + 5, \quad t > 0 \\ u(x, y, 0) = xy & 0 < x < 10, \quad \frac{x}{2} - 5 \le y \le -\frac{x}{2} + 5 \\ u(0, y, t) = 20 & -5 < y < 5, \quad t > 0 \\ u_x - 2u_y = 0 & 0 < x < 10, \quad y = \frac{x}{2} - 5, \quad t > 0 \\ u_x + 2u_y = 0 & 0 < x < 10, \quad y = -\frac{x}{2} + 5, \quad t > 0 \end{cases}$$

Exercise 18. The plate is an angular segment of a circular ring as in the figure (the angle is $\pi/4$). The sides are kept as indicated in the figure and the initial temperature in polar

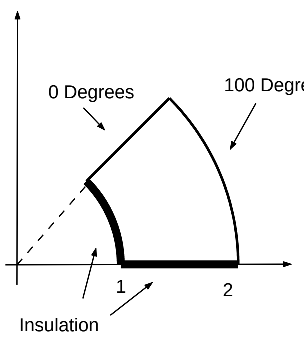

FIGURE 6. *Angular segment of a ring*

coordinates is $f(r, \theta) = r^2 \sin(4\theta)$.