NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS AND PROBLEMS IN HIGHER DIMENSIONS

1. Exercises.

In exercises 1 to 7 solve the nonhomogeneous boundary value problems

Exercise 1.

$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi,\ t > 0 \\ u(0,t) = 1,\ u(\pi,t) = 3 & t > 0 \\ u(x,0) = x & 0 < x < \pi \end{array}$$

First seek a steady state function $s(x)$ that satisfies the end points conditions. That is $s''(x) = 0$, $s(0) = 1$ and $s(\pi) = 3$. We find $s(x) = \dfrac{2x}{\pi} + 1$.

Now let $v(x,t) = u(x,t) - s(x)$. In order for u to solve the BVP, the function v must solve

$$\begin{array}{ll} v_t = v_{xx} & 0 < x < \pi,\ t > 0 \\ v(0,t) = 0,\ v(\pi,t) = 0 & t > 0 \\ v(x,0) = x - s(x) = \dfrac{\pi - 2}{\pi}x - 1 & 0 < x < \pi \end{array}$$

The v-problem can be solved by the method of separation of variables. We find

$$v(x,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 + (\pi - 3)(-1)^n}{n} \mathrm{e}^{-n^2 t} \sin(nx).$$

Therefore the solution u is given by

$$u(x,t) = s(x) + v(x,t) = \frac{2x}{\pi} + 1 - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 + (\pi - 3)(-1)^n}{n} \mathrm{e}^{-n^2 t} \sin(nx).$$

Exercise 2.

$$\begin{array}{ll} u_t = u_{xx} + \mathrm{e}^{-x} & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \end{array}$$

Exercise 3.

$$\begin{array}{ll} u_t = u_{xx} - x & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = x & 0 < x < \pi \end{array}$$

Seek the solution $u(x,t)$ as $u = v + w$ where the functions $v(x,t)$ and $w(x,t)$ are solutions of the BVPs

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) \\ v(0,t) = 0,\ v(\pi,t) = 0 \\ v(x,0) = x \end{cases} \quad \text{and} \quad \begin{cases} w_t(x,t) = w_{xx}(x,t) - x \\ w(0,t) = 0,\ w(\pi,t) = 0 \\ w(x,0) = 0 \end{cases}$$

The separation of variables gives

$$v(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{e}^{-n^2 t} \sin(nx).$$

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To solve the w-problem, we use the eigenfunctions expansion of the SL-problem $X'' + \lambda X = 0,\ X(0) = X(\pi) = 0$. That is, seek $w(x,t)$ as

$$w(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)\,.$$

where $c_n(t)$ are functions of t that need to be determined. Since $x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)$, the w-PDE can be rewritten as

$$\sum_{n=1}^{\infty} c_n'(t)\sin(nx) = -\sum_{n=1}^{\infty} n^2 c_n(t)\sin(nx) - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)\,.$$

The initial condition $w(x,0) = 0$ implies that $c_n(0) = 0$ for all $n \geq 1$. It follows that for $n \geq 1$, the function $c_n(t)$ satisfies the first order linear ODE problem

$$c_n'(t) + n^2 c_n(t) = \frac{2(-1)^n}{n}\,,\quad c_n(0) = 0.$$

We use the method of undetermined coefficients to find

$$c_n(t) = \frac{2(-1)^n}{n^3}\left(1 - \mathrm{e}^{-n^2 t}\right)\,.$$

Therefore

$$w(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3}\left(1 - \mathrm{e}^{-n^2 t}\right)\sin(nx)\,.$$

The solution u is:

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,\mathrm{e}^{-n^2 t}\sin(nx) + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3}\left(1 - \mathrm{e}^{-n^2 t}\right)\sin(nx) \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\left[1 - (1+n^2)\mathrm{e}^{-n^2 t}\right]\sin(nx) \end{aligned}$$

Exercise 4.

$$\begin{array}{ll} u_t = u_{xx} + 2t & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 100 & t > 0 \\ u(x,0) = 0 & 0 < x < \pi \end{array}$$

Exercise 5.

$$\begin{array}{ll} u_{tt} = u_{xx} - g & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0,\ u_t(x,0) = \sin x & 0 < x < \pi \end{array}$$

where g is a constant (gravitational for example).

Seek the solution $u(x,t)$ as $u = v + w$ where the functions $v(x,t)$ and $w(x,t)$ are solutions of the BVPs

$$\left\{\begin{array}{l} v_{tt}(x,t) = v_{xx}(x,t) \\ v(0,t) = 0,\ v(\pi,t) = 0 \\ v(x,0) = 0,\ v_t(x,0) = \sin x \end{array}\right. \quad\text{and}\quad \left\{\begin{array}{l} w_{tt}(x,t) = w_{xx}(x,t) - g \\ w(0,t) = 0,\ w(\pi,t) = 0 \\ w(x,0) = 0,\ w_t(x,0) = 0 \end{array}\right.$$

The separation of variables gives

$$v(x,t) = \sin t\,\sin x.$$

To solve the w-problem, we use the eigenfunctions expansion of the SL-problem $X'' + \lambda X = 0,\ X(0) = X(\pi) = 0$. That is, seek $w(x,t)$ as

$$w(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)\,.$$

where $c_n(t)$ are functions of t that need to be determined. Since $g = \dfrac{2g}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n}\sin(nx)$, the w-PDE can be rewritten as

$$\sum_{n=1}^{\infty} c_n''(t)\sin(nx) = -\sum_{n=1}^{\infty} n^2 c_n(t)\sin(nx) - \frac{2g}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(nx)\,.$$

The initial conditions $w(x,0) = 0$ and $w_t(x,0) = 0$ implies that $c_n(0) = 0$ and $c_n'(0) = 0$ for all $n \geq 1$. It follows that for $n \geq 1$, the function $c_n(t)$ satisfies the second order linear ODE problem

$$c_n''(t) + n^2 c_n(t) = \frac{2g\left[(-1)^n - 1\right]}{\pi n}\,, \quad c_n(0) = 0\,, \;\; c_n'(0) = 0.$$

We use the method of undetermined coefficients to find

$$c_n(t) = \frac{2g\left[(-1)^n - 1\right]}{\pi n^3}\left(1 - \cos(nt)\right)\,.$$

Therefore

$$w(x,t) = \sum_{n=1}^{\infty}\frac{2g\left[(-1)^n - 1\right]}{\pi n^3}\left(1 - \cos(nt)\right)\sin(nx)\,.$$

The solution u is:

$$u(x,t) = v(x,t) + w(x,t) = \sin t\,\sin x + \frac{2g}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^3}\left(1 - \cos(nt)\right)\sin(nx)\,.$$

Exercise 6.

$$\begin{array}{ll} u_{tt} = u_{xx} + \sin(2x) & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin x,\ u_t(x,0) = \sin(3x) & 0 < x < \pi \end{array}$$

Exercise 7.

$$\begin{array}{ll} u_{tt} = u_{xx} + \sin(2x)\cos t & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0,\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = 0,\ u_t(x,0) = \sin x & 0 < x < \pi \end{array}$$

Seek the solution $u(x,t)$ as $u = v + w$ where the functions $v(x,t)$ and $w(x,t)$ are solutions of the BVPs

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t) \\ v(0,t) = 0,\ v(\pi,t) = 0 \\ v(x,0) = 0,\ v_t(x,0) = \sin x \end{cases} \quad \text{and} \quad \begin{cases} w_{tt}(x,t) = w_{xx}(x,t) + \cos t\,\sin(2x) \\ w(0,t) = 0,\ w(\pi,t) = 0 \\ w(x,0) = 0,\ w_t(x,0) = 0 \end{cases}$$

The separation of variables gives

$$v(x,t) = \sin t\,\sin x.$$

To solve the w-problem, we use the eigenfunctions expansion of the SL-problem $X'' + \lambda X = 0,\ X(0) = X(\pi) = 0$. That is, seek $w(x,t)$ as

$$w(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)\,.$$

where $c_n(t)$ are functions of t that need to be determined. The w-PDE can be rewritten as

$$\sum_{n=1}^{\infty} c_n''(t)\sin(nx) = -\sum_{n=1}^{\infty} n^2 c_n(t)\sin(nx) + \cos t\,\sin(2x)\,.$$

The initial conditions $w(x,0) = 0$ and $w_t(x,0) = 0$ implies that $c_n(0) = 0$ and $c_n'(0) = 0$ for all $n \geq 1$. It follows that for $n \neq 2$, the function $c_n(t)$ satisfies the second order linear ODE problem

$$\begin{cases} c_n''(t) + n^2 c_n(t) = 0 \\ c_n(0) = 0\,, \quad c_n'(0) = 0 \end{cases} \quad \Longrightarrow \quad c_n(t) = 0$$

For $n = 2$, $c_2(t)$ satisfies the ODE problem

$$c_2''(t) + 4c_2(t) = \cos t, \quad c_2(0) = 0, \; c_2'(0) = 0.$$

We use the method of undetermined coefficients to find

$$c_2(t) = \frac{\cos t - \cos(2t)}{3}\,.$$

Therefore

$$w(x,t) = \frac{\cos t - \cos(2t)}{3}\,\sin(2x)\,.$$

The solution u is:

$$u(x,t) = v(x,t) + w(x,t) = \sin t\,\sin x + \frac{\cos t - \cos(2t)}{3}\,\sin(2x)\,.$$

Exercise 8. The function $f(x,y)$ is doubly periodic with period 2π in x and in y. It is given on $[-\pi,\ \pi]^2$ by $f(x,y) = xy^2$. Find the double Fourier series of f.

Exercise 9. Same question as in problem 8 for the function given on $[-\pi,\ \pi]^2$ by $f(x,y) = x^2y^2$.

The function $f(x,y)$ which is 2π-periodic in x and 2π-periodic in y and defined in the square $[-\pi,\ \pi]^2$ by $f(x,y) = x^2y^2$ is even in x and even in y. Therefore its double Fourier series has the form

$$\frac{A_{00}}{4} + \frac{1}{2}\sum_{n=1}^{\infty} A_{n,0}\cos(nx) + \frac{1}{2}\sum_{m=1}^{\infty} A_{0m}\cos(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos(nx)\,\cos(my)$$

with

$$A_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} x^2y^2\cos(nx)\cos(my)\,dx\,dy.$$

We have

$$A_{00} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} x^2y^2\,dx\,dy = \frac{4\pi^4}{9}\,.$$

A repeated integration by parts shows that

$$\int_0^{\pi} x^2\cos(nx)\,dx = \left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_0^{\pi} = \frac{2\pi(-1)^n}{n^2}\,.$$

It follows that

$$A_{n,0} = \frac{8\pi^2(-1)^n}{3n^2}\,, \quad A_{0,m} = \frac{8\pi^2(-1)^m}{3m^2}\,,$$

and for $n, m \geq 1$

$$A_{n,m} = \frac{16(-1)^{n+m}}{n^2m^2}\,.$$

Hence for $-\pi \leq x \leq \pi,\ \ -\pi \leq y \leq \pi$, we have

$$x^2y^2 = \frac{\pi^4}{9}+\sum_{n=1}^{\infty}\frac{4\pi^2(-1)^n}{3n^2}\cos(nx)+\sum_{m=1}^{\infty}\frac{4\pi^2(-1)^m}{3m^2}\cos(my)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{16(-1)^{n+m}}{n^2m^2}\cos(nx)\cos(my)\,.$$

Exercise 10. Let $f(x,y) = 1$ on the square $[0,\ 1]^2$. Find

1. The Fourier cosine-cosine series of f.
2. The Fourier cosine-sine series of f.
3. The Fourier sine-sine series of f.
4. The Fourier sine-cosine series of f.

Exercise 11. Same questions as in problem 10 for the function $f(x,y) = xy$ on the square $[0,\ \pi]^2$.

(1) Fourier cosine-cosine series:

$$xy = \frac{A_{00}}{4} + \frac{1}{2}\sum_{n=1}^{\infty}A_{n,0}\cos(nx) + \frac{1}{2}\sum_{m=1}^{\infty}A_{0m}\cos(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\cos(nx)\cos(my)$$

with

$$A_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy\cos(nx)\cos(my)\,dx\,dy.$$

We have

$$A_{00} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy\,dx\,dy = \pi^2\,.$$

Integration by parts shows that

$$\int_0^{\pi} x\cos(nx)\,dx = \left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}\,.$$

It follows that

$$A_{n,0} = 2\frac{(-1)^n - 1}{n^2}\,,\ \ A_{0,m} = 2\frac{(-1)^m - 1}{m^2}\,,$$

and for $n, m \geq 1$,

$$A_{n,m} = \frac{4}{\pi^2}\,\frac{(-1)^n - 1}{n^2}\,\frac{(-1)^m - 1}{m^2}\,.$$

Hence for $0 \leq x \leq \pi,\ \ 0 \leq y \leq \pi$, we have

$$\begin{aligned} xy \ &= \frac{\pi^2}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx) + \sum_{m=1}^{\infty}\frac{(-1)^m - 1}{m^2}\cos(my) + \\ &\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{4[(-1)^n - 1][(-1)^m - 1]}{n^2m^2}\cos(nx)\cos(my)\,. \end{aligned}$$

(2) Fourier cosine-sine series:

$$xy = \frac{1}{2}\sum_{m=1}^{\infty}B_{0,m}\sin(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nm}\cos(nx)\sin(my)$$

with

$$B_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy\cos(nx)\sin(my)\,dx\,dy.$$

We have

$$B_{0m} = \frac{4}{\pi^2}\left(\int_0^{\pi} x\,dx\right)\left(\int_0^{\pi} y\sin(my)\,dy\right) = \frac{2\pi(-1)^{m+1}}{m}\,,$$

and for $n, m \geq 1$,

$$B_{n,m} = \frac{4}{\pi^2}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^\pi \left[-\frac{y\cos(my)}{m} + \frac{\sin(my)}{m^2}\right]_0^\pi$$

$$= \frac{4}{\pi}\frac{(-1)^n - 1}{n^2}\frac{(-1)^{m+1}}{m}.$$

Hence for $0 \leq x \leq \pi,\ \ 0 \leq y \leq \pi$, we have

$$xy = \pi\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(my) + \frac{4}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n - 1}{n^2}\frac{(-1)^{m+1}}{m}\cos(nx)\ \sin(my)$$

(3) Fourier sine-sine series:

$$xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin(nx)\ \sin(my)$$

with

$$B_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\cos(nx)\sin(my)\,dx\,dy.$$

$$B_{n,m} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\sin(nx)\ \sin(my)\,dxdy = 4\frac{(-1)^{n+m}}{nm}.$$

Hence for $0 \leq x \leq \pi,\ \ 0 \leq y \leq \pi$, we have

$$xy = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}}{nm}\sin(nx)\ \sin(my)$$

(4) Fourier sine-cosine series: For $0 \leq x \leq \pi,\ \ 0 \leq y \leq \pi$, we have

$$xy = \pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(ny) + \frac{4}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{(-1)^m - 1}{m^2}\sin(nx)\ \cos(my).$$

Exercise 12. Find the Fourier sine-sine series of the function $f(x, y)$ given on the square $[0,\ \pi]^2$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x > y \end{cases}$$

In the remaining exercises use multiple Fourier series to solve the BVP (double series except in the last exercise where you can use triple Fourier series).

Exercise 13.

$$\begin{array}{ll} u_t = 4(u_{xx} + u_{yy}), & 0 < x < 2,\ 0 < y < 1,\ t > 0 \\ u_x(0, y, t) = u_x(2, y, t) = 0, & 0 < y < 1,\ t > 0 \\ u(x, 0, t) = u(x, 1, t) = 0, & 0 < x < 2,\ t > 0 \\ u(x, y, 0) = 100 & 0 < x < 2,\ 0 < y < 1\,. \end{array}$$

If $u(x, y, t) = X(x)Y(y)T(t)$ is a nontrivial solution the homogeneous part of the BVP, then the functions X, Y, and T solve the ODE problems:

$$\begin{cases} X''(x) + \alpha X(x) = 0, \\ X'(0) = 0, \quad X'(2) = 0 \end{cases} \qquad \begin{cases} Y''(y) + \beta Y(y) = 0, \\ Y(0) = 0, \quad Y(1) = 0 \end{cases} \qquad T'(t) + 4\lambda T(t) = 0$$

where α, β, λ are separation constants and $\lambda = \alpha + \beta$.

The eigenvalues and eigenfunctions of the X-problem are:

$$\alpha_n = \left(\frac{n\pi}{2}\right)^2, \quad X_n(x) = \cos\frac{n\pi x}{2}, \quad n = 0,\ 1,\ 2,\ 3,\ \cdots$$

The eigenvalues and eigenfunctions of the Y-problem are:

$$\beta_m = (m\pi)^2, \quad Y_m(y) = \sin(m\pi y), \quad m = 1,\ 2,\ 3,\ \cdots$$

For each pair of integers n, m, we have $\lambda_{nm} = \dfrac{\pi^2(n^2+4m^2)}{4}$ and an independent solution of the T-problem is $T_{nm}(t) = \mathrm{e}^{-4\lambda_{nm}t}$. The solutions with separated variables of the homogeneous part are

$$\mathrm{e}^{-\pi^2(n^2+4m^2)\,t}\cos\frac{n\pi x}{2}\,\sin(m\pi y)\,.$$

The series representation of the general solution is

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\mathrm{e}^{-\pi^2(n^2+4m^2)\,t}\cos\frac{n\pi x}{2}\,\sin(m\pi y)\,.$$

We find the constants C_{nm} so that u solves the complete BVP by using the nonhomogeneous condition

$$u(x,y,0) = 100 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\frac{n\pi x}{2}\,\sin(m\pi y)\,.$$

The last series is therefore the Fourier cosine-sine series of the function $f(x,y) = 100$. We have

$$C_{0m} = \frac{1}{4}\int_0^2\int_0^1 100\sin(m\pi y)\,dxdy = \frac{50\,[(-1)^m - 1]}{\pi m}$$

and for $n \geq 1$

$$C_{nm} = \frac{1}{2}\int_0^2\int_0^1 100\cos\frac{n\pi x}{2}\,\sin(m\pi y)\,dxdy = 0.$$

Therefore the solution of the BVP is

$$u(x,y,t) = \frac{50}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m - 1}{m}\mathrm{e}^{-4\pi^2m^2\,t}\sin(m\pi y)\,.$$

Note that the solution is independent on x.

Exercise 14.

$$\begin{array}{ll}
u_{tt} = u_{xx} + u_{yy}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\
u(0,y,t) = u(\pi,y,t) = 0, & 0 < y < \pi,\ t > 0 \\
u(x,0,t) = u(x,\pi,t) = 0, & 0 < x < \pi,\ t > 0 \\
u(x,y,0) = 0.05x(\pi - x)y(\pi - y) & 0 < x < \pi,\ 0 < y < \pi \\
u_t(x,y,0) = 0 & 0 < x < \pi,\ 0 < y < \pi\ .
\end{array}$$

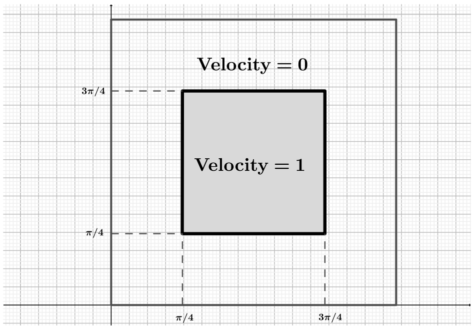
Exercise 15.

$$\begin{array}{ll}
u_{tt} = u_{xx} + u_{yy}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\
u(0,y,t) = u(\pi,y,t) = 0, & 0 < y < \pi,\ t > 0 \\
u(x,0,t) = u(x,\pi,t) = 0, & 0 < x < \pi,\ t > 0 \\
u(x,y,0) = 0 & 0 < x < \pi,\ 0 < y < \pi \\
u_t(x,y,0) = f(x,y) & 0 < x < \pi,\ 0 < y < \pi\ .
\end{array}$$

where

$$f(x,y) = \begin{cases} 1 & \text{if } \ \pi/4 < x < 3\pi/4,\ \pi/4 < y < 3\pi/4 \\ 0 & \text{elsewhere} \end{cases}$$

This problem models the vibrations of a struck square membrane. The initial velocity is $f(x,y) = 1$ in the middle square $[\pi/4,\ \ 3\pi/4]^2$ and zero elsewhere. If $u(x,y,t) = X(x)Y(y)T(t)$



is a nontrivial solution the homogeneous part of the BVP, then the functions X, Y, and T solve the ODE problems:

$$\begin{cases} X''(x) + \alpha X(x) = 0, \\ X(0) = 0,\ X(\pi) = 0 \end{cases} \quad \begin{cases} Y''(y) + \beta Y(y) = 0, \\ Y(0) = 0,\ Y(\pi) = 0 \end{cases} \quad \begin{cases} T''(t) + \lambda T(t) = 0 \\ T(0) = 0 \end{cases}$$

where α, β, λ are separation constants and $\lambda = \alpha + \beta$.
The eigenvalues and eigenfunctions of the X-problem are:

$$\alpha_n = n^2, \quad X_n(x) = \sin(nx), \quad n = 1,\, 2,\, 3,\, \cdots$$

The eigenvalues and eigenfunctions of the Y-problem are:

$$\beta_m = m^2, \quad Y_m(y) = \sin(my), \quad m = 1,\, 2,\, 3,\, \cdots$$

For each pair of integers n, m, we have $\lambda_{nm} = \omega_{nm}^2$ with $\omega_{nm} = \sqrt{n^2 + m^2}$ and an independent solution of the T-problem is $T_{nm}(t) = \sin(\omega_{nm} t)$. The solutions with separated variables of the homogeneous part are

$$\sin(\omega_{nm} t)\, \sin(nx)\, \sin(my)\,.$$

The series representation of the general solution is

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin(\omega_{nm} t)\, \sin(nx)\, \sin(my)\,.$$

To find the constants C_{nm} so that u solves the complete BVP we start by computing $u_t(x, y, t)$

$$u_t(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} C_{nm} \cos(\omega_{nm} t)\, \sin(nx)\, \sin(my)$$

and then evaluate at $t = 0$.

$$u_t(x, y, 0) = f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} C_{nm}\, \sin(nx)\, \sin(my)\,.$$

The last series is therefore the Fourier sine-sine series of the function $f(x, y)$. We have

$$\begin{aligned} \omega_{nm} C_{nm} &= \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x, y)\, \sin(nx)\, \sin(my)\, dxdy = 0 \\ &= \frac{4}{\pi^2} \int_{\pi/4}^{3\pi/4} \sin(nx) dx \int_{\pi/4}^{3\pi/4} \sin(my) dy \\ &= \frac{4}{\pi^2 n\, m} \left[\cos\frac{3n\pi}{4} - \cos\frac{n\pi}{4} \right] \left[\cos\frac{3m\pi}{4} - \cos\frac{m\pi}{4} \right] \end{aligned}$$

If we use the trigonometric identity $\cos A - \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$, we can rewrite

$$\omega_{nm} C_{nm} = \frac{16}{\pi^2 n\, m} \cos\frac{n\pi}{2}\cos\frac{n\pi}{4}\cos\frac{m\pi}{2}\cos\frac{m\pi}{4}\,.$$

Since $\cos\frac{N\pi}{2} = 0$ if N is odd and $\cos\frac{N\pi}{2} = (-1)^J$ if $N = 2J$, it follows that $C_{nm} = 0$ if either n or m is not a multiple of 4 and

$$c_{4j,4k} = \frac{(-1)^{j+k}}{\pi^2 4\sqrt{j^2+k^2}\, j\, k}\,.$$

Therefore the solution of the BVP is

$$u(x,y,t) = \frac{1}{4\pi^2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{j+k}}{\sqrt{j^2+k^2}\, j\, k}\sin\left[4\sqrt{j^2+k^2}\, t\right]\sin(4jx)\sin(4ky)\,.$$

Note that the solution is independent on x.

Exercise 16.

$$\begin{array}{ll} u_{xx}+u_{yy} = 2u+1, & 0<x<\pi,\quad 0<y<\pi,\\ u(0,y)=u(\pi,y)=0, & 0<y<\pi,\\ u(x,0)=u(x,\pi)=0, & 0<x<\pi\,. \end{array}$$

Exercise 17.

$$\begin{array}{ll} u_{xx}+u_{yy} = xy, & 0<x<\pi,\quad 0<y<\pi,\\ u(0,y)=u(\pi,y)=0, & 0<y<\pi,\\ u(x,0)=u(x,\pi)=0, & 0<x<\pi\,. \end{array}$$

The unique solution $u(x,y)$ can be computed using three approaches.

(1) Expansion with respect to eigenfunctions $\sin(nx)$:We seek the solution

$$u(x,y) = \sum_{n=1}^{\infty} c_n(y)\sin(nx)\,.$$

First expand xy in its Fourier sine series in x:

$$xy = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}y}{n}\sin(nx)\,.$$

The PDE becomes

$$\sum_{n=1}^{\infty}\left[c_n''(y) - n^2 c_n(y)\right]\sin(nx) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}y}{n}\sin(nx)\,.$$

The functions $c_n(y)$ satisfies the ODE problem

$$c_n''(y) - n^2 c_n(y = \frac{2(-1)^{n+1}y}{n},\quad c_n(0)=0,\quad c_n(\pi)=0\,.$$

By using the UC method we find $c_n(y) = \frac{2(-1)^n\pi}{n^3}\left[\frac{\sinh(ny)}{\sinh(n\pi)} - \frac{y}{\pi}\right]$. The solution of the BVP is

$$u(x,y) = \sum_{n=1}^{\infty}\frac{2(-1)^n\pi}{n^3}\left[\frac{\sinh(ny)}{\sinh(n\pi)} - \frac{y}{\pi}\right]\sin(nx)\,.$$

(2) Expansion with respect to eigenfunctions $\sin(my)$:We seek the solution

$$u(x,y) = \sum_{m=1}^{\infty} c_n(x)\sin(my)\,.$$

A similar arguments as above give the solution of the BVP as

$$u(x,y) = \sum_{m=1}^{\infty} \frac{2(-1)^m\pi}{m^3}\left[\frac{\sinh(mx)}{\sinh(m\pi)} - \frac{x}{\pi}\right]\sin(my)\,.$$

(3) Expansion with respect to eigenfunctions $\sin(nx)\sin(my)$:We seek the solution

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin(nx)\,\sin(my)\,.$$

First expand xy into its Fourier sine-sine series

$$xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4(-1)^{n+m}}{n\,m}\,\sin(nx)\,\sin(my)\,.$$

The PDE becomes

$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n^2+m^2)C_{nm}\sin(nx)\,\sin(my) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4(-1)^{n+m}}{n\,m}\,\sin(nx)\,\sin(my)\,.$$

It follows that $C_{nm} = \dfrac{4(-1)^{n+m+1}}{(n^2+m^2)\,n\,m}$ and the solution of the BVP is

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4(-1)^{n+m+1}}{(n^2+m^2)\,n\,m}\,\sin(nx)\,\sin(my)\,.$$

Exercise 18. (Dirichlet problem in a cube)

$$\begin{array}{ll}
u_{xx}+u_{yy}+u_{zz}=0, & 0<x<\pi,\quad 0<y<\pi,\quad 0<z<\pi,\\
u(0,y,z)=u(\pi,y,z)=0, & 0<y<\pi,\quad 0<z<\pi\\
u(x,0,z)=-\sin(2x)\sin(5z), & 0<x<\pi\quad 0<z<\pi,\\
u(x,\pi,z)=\sin(3x)\sin(z), & 0<x<\pi\quad 0<z<\pi,\\
u(x,y,0)=\sin x\sin(2y),\quad u(x,y,\pi)=0, & 0<x<\pi,\quad 0<y<\pi\,.
\end{array}$$