BESSEL EQUATIONS AND BESSEL FUNCTIONS

1. Exercises

Exercise 1. The table bellow lists approximate values of the Gamma function for values of x in the interval $[0,\ 1]$. Use the table together with the fundamental property of the Gamma function to find the following values

$$\Gamma(5.45),\quad \Gamma(3.10),\quad \Gamma(6.05),\quad \Gamma(4.85),$$
$$\Gamma(-0.75),\quad \Gamma(-4.65),\quad \Gamma(-0.01),\quad \Gamma(-2,85),\quad \Gamma(-3.75).$$

x	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\Gamma(x)$	19.470	9.513	6.220	4.591	3.626	2.992	2.546	2.218	1.968	1.773
x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
$\Gamma(x)$	1.616	1.489	1.385	1.298	1.225	1.164	1.113	1.069	1.032	1.00

We use repeatedly the property $\Gamma(x+1) = x\Gamma(x)$ to obtain

$$\begin{aligned}
\Gamma(5.45) &= 4.45\,\Gamma(4.45) = (4.45)\cdot(3.45)\cdot(2.45)\cdot(1.45)\cdot(0.45)\cdot\Gamma(0.45)\\
&= (4.45)\cdot(3.45)\cdot(2.45)\cdot(1.45)\cdot(0.45)\cdot(1.968)\\
&= 43.300
\end{aligned}$$

$$\begin{aligned}
\Gamma(3.10) &= (2.10)\cdot(1.10)\cdot(0.10)\Gamma(0.10)\\
&= (2.10)\cdot(1.10)\cdot(0.10)\cdot(9.513) = 2.197
\end{aligned}$$

$$\begin{aligned}
\Gamma(6.05) &= (5.05)\cdot(4.05)\cdot(3.05)\cdot(2.05)\cdot(1.05)\cdot(0.05)\Gamma(0.05)\\
&= (5.05)\cdot(4.05)\cdot(3.05)\cdot(2.05)\cdot(1.05)\cdot(0.05)\cdot(19.470)\\
&= 130.71
\end{aligned}$$

For $x < 0$, we use repeatedly $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$ to obtain

$$\begin{aligned}
\Gamma(-4.65) &= \frac{\Gamma(-3.65)}{(-4.65)} = \frac{\Gamma(-2.65)}{(-4.65)\cdot(-3.65)}\\
&= \frac{\Gamma(-0.65)}{(-4.65)\cdot(-3.65)\cdot(-2.65)\cdot(-1.65)}\\
&= \frac{\Gamma(0.35)}{(-4.65)\cdot(-3.65)\cdot(-2.65)\cdot(-1.65)\cdot(-0.65)}\\
&= -\frac{2.546}{(4.65)\cdot(3.65)\cdot(2.65)\cdot(1.65)\cdot(0.65)}\\
&= -0.053
\end{aligned}$$

Exercise 2. The aim of this exercise is to establish the formulas

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta d\theta \qquad x>0,\ y>0 \qquad (*)$$

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1. Show that

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-s}s^{x-1}ds \int_0^\infty e^{-t}t^{y-1}dt = \int_0^\infty \int_0^\infty e^{-(u^2+v^2)}u^{2x-1}v^{2y-1}dudv$$

(*Hint*: consider the substitutions $s = u^2$ and $t = v^2$)

By using the substitution $s = u^2$ we get

$$\Gamma(x) = \int_0^\infty e^{-s}s^{x-1}ds = \int_0^\infty e^{-u^2}u^{2x-2}\,2udu = 2\int_0^\infty e^{-u^2}u^{2x-1}\,du.$$

Similarly,

$$\Gamma(y) = 2\int_0^\infty e^{-s}s^{x-1}ds = 2\int_0^\infty e^{-v^2}v^{2y-1}\,dv.$$

It follows that

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}u^{2x-1}v^{2y-1}\,dudv.$$

2. Use polar coordinates $u = r\cos\theta$, $v = r\sin\theta$ to establish formula $(*)$.

With the use of polar coordinate, the region of integration is the first quadrant defined by $0 < r < \infty$ and $0 < \theta < \pi/2$. Using $u^2 + v^2 = r^2$ and $dudv = rdrd\theta$ we get

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}u^{2x-1}v^{2y-1}\,dudv \\ &= 4\int_0^\infty \int_0^{\pi/2} e^{-r^2}(r\cos\theta)^{2x-1}(r\sin\theta)^{2y-1}\,rdrd\theta \\ &= 4\int_0^\infty e^{-r^2}r^{2x+2y-2}\,rdr \int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta \\ &= 2\int_0^\infty e^{-t}t^{x+y-1}\,dt \int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta \\ &= 2\Gamma(x+y)\int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta\end{aligned}$$

3. Use formula $(*)$ to establish the following formula $(j, k \in \mathbb{Z}^+)$

$$\int_0^{\pi/2} \cos^{2j-1}\theta\sin^{2k-1}\theta d\theta = \frac{(j-1)!\,(k-1)!}{2\,(k+j-1)!}$$

$$\int_0^{\pi/2} \cos^{2j-1}\theta\sin^{2k-1}\theta d\theta = \frac{\Gamma(j)\Gamma(k)}{2\Gamma(j+k)} = \frac{(j-1)!\,(k-1)!}{2\,(k+j-1)!}$$

4. Use formula $(*)$ together with $\Gamma(j + (1/2)) = \dfrac{(2j-1)!}{2^{2j-1}(j-1)!}\sqrt{\pi}$. to establish

$$\int_0^{\pi/2} \cos^{2j}\theta\sin^{2k-1}\theta d\theta = \frac{2^{2k-1}(2j-1)!(k-1)!(k+j-1)!}{(j-1)!(2k+2j-1)!}$$

(*Hint*: Use $x = j + (1/2)$ and $y = k$ in formula $(*)$.)

5. Use the table of values of the Gamma function given in exercise 1 to find an approximation of the integral

$$\int_0^{\pi/2} \cos^{\pi}\theta\sin^{e}\theta d\theta$$

$$\int_0^{\pi/2} \cos^\pi \theta \sin^{\mathrm{e}} \theta d\theta = \frac{\Gamma\left(\frac{\pi+1}{2}\right)\Gamma\left(\frac{\mathrm{e}+1}{2}\right)}{2\Gamma\left(\frac{\pi+\mathrm{e}}{2}+1\right)} \approx \frac{\Gamma(2.07)\Gamma(1.86)}{2\Gamma(3.93)}$$

We have $\Gamma(2.07) \approx 1.46$, $\Gamma(1.86) \approx 0.96$ and $\Gamma(3.93) \approx 5.43$. Therefore

$$\int_0^{\pi/2} \cos^\pi \theta \sin^{\mathrm{e}} \theta \, d\theta \approx \frac{(1.46)\cdot(0.96)}{2\cdot(5.48)} \approx 0.13$$

Exercise 3. The Psi function is defined as the logarithmic derivative of Γ:

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

Use the fundamental property of Γ to show that Ψ satisfies

$$\Psi(x+1) = \Psi(x) + \frac{1}{x}\,.$$

Exercise 4. Write the first five terms of the series representation of J_0; J_1; J_2; J_{-3}; $J_{3/4}$; $J_{1/5}$.

We have

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1+\alpha)} \left(\frac{x}{2}\right)^{2j+\alpha}\,.$$

Therefore

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,j!} \left(\frac{x}{2}\right)^{2j}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{64} - \frac{x^6}{1152} + \frac{x^8}{147456} + \cdots$$

$$J_1(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!} \left(\frac{x}{2}\right)^{2j+1}$$

$$= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} + \frac{x^9}{1474560} + \cdots$$

$$J_{-3}(x) = (-1)^3 J_3(x) = -\sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+3)!} \left(\frac{x}{2}\right)^{2j+3}$$

$$= -\frac{x^3}{48} + \frac{x^5}{768} - \frac{x^7}{30720} + \frac{x^9}{2211840} - \frac{x^11}{495452160} \cdots$$

$$J_{1/5}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+4/5)!} \left(\frac{x}{2}\right)^{2j+1/5}$$

$$= \frac{1}{\Gamma(1/5)} \sqrt[5]{\frac{x}{2}} \left[5 - \frac{5^2}{6\cdot 1}\left(\frac{x}{2}\right)^2 + \frac{5^3}{2!\cdot 11\cdot 6\cdot 1}\left(\frac{x}{2}\right)^4 - \frac{5^4}{3!\cdot 16\cdot 11\cdot 6\cdot 1}\left(\frac{x}{2}\right)^6 + \frac{5^5}{4!\cdot 21\cdot 16\cdot 11\cdot 6\cdot 1}\left(\frac{x}{2}\right)^8\right] + \cdots$$

Exercise 5. Use the series expansion of $J_{-1/2}$ to establish

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \ .$$

You can also establish this formula by using property (5) with $\alpha = 1/2$ and $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

We will use the facts that $\cos x = \sum j = 0^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}$ and $\Gamma(j + (1/2)) = \frac{(2j-1)!}{2^{2j-1}(j-1)!}\sqrt{\pi}$. We have

$$J_{-1/2}(x) \quad = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1/2)} \left(\frac{x}{2}\right)^{2j-1/2} = \sqrt{\frac{2}{x}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1/2)} \left(\frac{x}{2}\right)^{2j}$$

Since

$$j!\,\Gamma(j+1/2) = \frac{(2j-1)!\,\sqrt{\pi}\,j!}{2^{2j-1}(j-1)!} = \sqrt{\pi}\frac{(2j-1)!\,(2j)}{2^{2j}} = \sqrt{\pi}\frac{(2j)!}{2^{2j}} \,,$$

then

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j}}{\sqrt{\pi}\,(2j)!} \left(\frac{x}{2}\right)^{2j} = \sqrt{\frac{2}{\pi\, x}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} = \sqrt{\frac{2}{\pi\, x}} \ \cos x \,.$$

Alternative solution. Use Property (5) with $\alpha = 1/2$: $\frac{d}{dx}\left(x^{1/2} J_{1/2}(x)\right) = x^{1/2} J_{-1/2}(x)$. Since $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, then

$$\frac{d}{dx}\left(x^{1/2} J_{1/2}(x)\right) = \frac{d}{dx}\left(\sqrt{\frac{2}{\pi}} \ \sin x\right) = \sqrt{\frac{2}{\pi}} \ \cos x$$

and the formula $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ follows.

Exercise 6. Repeat the steps of example 1 to show that

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x\right) \ .$$

Exercise 7. Find the expressions of $J_{5/2}$ and of $J_{-5/2}$.

To find $J_{5/2}$, use the property property $J_{\alpha+1}(x) + J_{\alpha-1}(x) = \frac{2\alpha}{x} J_\alpha(x)$ with $\alpha = 3/2$. That is, $J_{5/2}(x) + J_{1/2}(x) = \frac{3}{x} J_{3/2}(x)$. This gives

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$$

The above property with $\alpha = 1/2$ leads to

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

Hence

$$J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2}(x) = \left[\frac{3}{x^2} - 1\right] J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x).$$

Since $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$, then

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3-x^2}{x^2} \sin x - \frac{3}{x} \cos x \,. \right]$$

Exercise 8. Use the table of values of J_0 and J_1 to find the following values

$$J_2(.5), \quad J_3(5), \quad J_4(8.5)$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$J_0(x)$	1.0000	0.9385	0.7652	0.5118	0.2239	-0.0484	-0.2601
$J_1(x)$	0	0.2423	0.4401	0.5579	0.5767	0.4971	0.3391
x	3.5	4	4.5	5	5.5	6	6.5
$J_0(x)$	-0.3801	-0.3971	-0.3205	-0.1776	-0.0068	0.1506	0.2601
$J_1(x)$	0.1374	-0.0660	-0.2311	-0.3276	-0.3414	-0.2767	-0.1538
x	7	7.5	8	8.5	9	9.5	10
$J_0(x)$	0.3001	0.2663	0.1717	0.0419	-0.0903	-0.1939	-0.2459
$J_1(x)$	-0.0047	0.1352	0.2346	0.2731	0.2453	0.1613	0.0435

To find $J_4(8.5)$, we start by using the property $J_{\alpha+1}(x) = \frac{2\alpha}{x} J_\alpha(x) - J_{\alpha-1}(x)$ with $\alpha = 3$ so that

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x).$$

Then the same property for $\alpha = 1,\ 2$ gives

$$J_2(x) \quad = \frac{2}{x} J_1(x) - J_0(x)$$

$$J_3(x) \quad = \frac{4}{x} J_2(x) - J_1(x) = \frac{8-x^2}{x^2} J_1(x) - \frac{4}{x} J_0(x)$$

After replacing these expressions of J_3 and J_2 in the expression for J_4 we obtain

$$J_4(x) = \frac{8(6-x^2)}{x^3} J_1(x) - \frac{24-x^2}{x^2} J_0(x)\,.$$

Therefore, by using $J_0(8.5) \approx 0.0419$ and $J_1(8.5) \approx 0.2731$ (from the table), we obtain

$$J_4(8.5) \approx \frac{8(6-8.5^2)}{8.5^3} 0.2731 - \frac{24-8.5^2}{8.5^2} 0.0419 \approx -0.2077\,.$$

Exercise 9. Prove that $\int_0^x s J_0(s) ds = x J_1(x)$.

Exercise 10. Find the integrals

$$\int x^9 J_8(x) dx, \quad \int x^{-3/2} J_{5/2}(x) dx, \quad \int x^5 J_2(x) dx$$

We use the properties

$$\int x^{-\alpha} J_{\alpha+1}(x)\, dx = -x^{-\alpha} J_\alpha(x) + C \quad \text{and} \quad \int x^\alpha J_{\alpha-1}(x)\, dx = x^\alpha J_\alpha(x) + C$$

We have

$$\int x^{-3/2} J_{5/2}(x) dx = -x^{-3/2} J_{3/2}(x) + C.$$

and

$$\begin{aligned}\int x^5 J_2(x)dx &= \int x^2(x^3 J_2(x))dx = \int x^2(x^3 J_3(x))'dx \\ &= x^2(x^3 J_3(x)) - 2\int x(x^3 J_3(x))dx = x^5 J_3(x) - 2\int x^4 J_3(x)dx \\ &= x^5 J_3(x) - 2x^4 J_4(x) + C\end{aligned}$$

Exercise 11. Find the integrals

$$\int x^{2-\alpha} J_{\alpha+1}(x)dx, \quad \int J_1(x)dx, \qquad \int (J_2(x) - J_0(x))\,dx$$

To find $\int J_2(x)dx$, use properties

$$2\alpha J_\alpha(x) = J_{\alpha+1}(x) + J_{\alpha-1}(x) \quad \text{and} \quad J_{\alpha+1}(x) = -x^\alpha(x^{-\alpha}J_\alpha(x))'$$

so that

$$\begin{aligned}4\int J_2(x)dx &= \int (xJ_3(x) + xJ_1(x))dx = \int x(-x^2(x^{-2}J_2(x))')\,dx + \int x(-J_0(x)')\,dx \\ &= -\int x^3\,(x^{-2}J_2(x))')\,dx - \int x\,J_0(x)'\,dx \\ &= -xJ_2(x) + 3\int J_2(x)dx - xJ_0(x) + \int J_0(x)dx\end{aligned}$$

It follows that

$$\int (J_2(x) - J_0(x))\,dx = -xJ_2(x) - xJ_0(x) + C\,.$$

We have

$$\begin{aligned}\int x^{2-\alpha} J_{\alpha+1}(x)dx &= -\int x^2\left(x^{-\alpha}J_\alpha(x)\right)'\,dx \\ &= -x^2\left(x^{-\alpha}J_\alpha(x)\right) + 2\int x^{1-\alpha}J_\alpha(x)\,dx \\ &= -x^{2-\alpha}J_\alpha(x) - 2\int \left(x^{-(\alpha-1)}J_{\alpha-1}(x)\right)'\,dx \\ &= -x^{2-\alpha}J_\alpha(x) - 2x^{1-\alpha}J_{\alpha-1}(x) + C\end{aligned}$$

Exercise 12. Find the integrals

$$\int [J_3(x) - J_5(x)]\,dx, \quad \int_0^x s^4 J_1(s)ds$$

$$\begin{aligned}10\int J_5(x)dx &= \int (xJ_6(x)+xJ_4(x))dx \\ &= \int x\left(-x^5(x^{-5}J_5(x))'\right)dx + \int x\left(-x^3(x^{-3}J_3(x))'\right)dx \\ &= -xJ_5(x)+6\int J_5(x)dx - xJ_3(x) + 4\int J_3(x)dx\end{aligned}$$

It follows that

$$\int [J_3(x)-J_5(x)]\,dx = \frac{x}{4}[J_5(x)+J_3(x)]+C.$$

We have

$$\begin{aligned}\int s^4J_1(s)ds &= \int s^4(-J_0(s))'ds = -s^4J_0(s)+4\int s^3J_1(s)ds \\ &= -s^4J_0(s)+4\int s^2(sJ_1(s))'ds = -s^4J_0(s)+4s^3J_1(s)-8\int s^2J_1(s)ds \\ &= -s^4J_0(s)+4s^3J_1(s)-8\int\left(s^2J_2(s)\right)'ds = \\ &= -s^4J_0(s)+4s^3J_1(s)-8s^2J_2(s)+C\end{aligned}$$

Exercise 13. Show that

$$\int_0^R x^\alpha J_{\alpha-1}(\lambda x)dx = \frac{R^\alpha}{\lambda}J_\alpha(\lambda R)$$

Exercise 14. Show that

$$x^2J_\alpha''(x)-(\alpha^2-x^2-\alpha x^2)J_\alpha(x)-xJ_{\alpha+1}(x)=0$$

(*Hint*: Use Bessel's equation and property 4)

Exercise 15. Show that

$$\int_0^x J_3(s)ds = 1-J_2(x)-2\frac{J_1(x)}{x}$$

(*Hint*: Start with $J_3(s)=s^2(s^{-2}J_3(s))$ and use integration by parts)

We have

$$\begin{aligned}\int J_3(s)ds &= \int s^2(s^{-2}J_3(s))ds = -\int s^2(s^{-2}J_2(s))'ds \\ &= -J_2(s)+2\int s^{-1}J_2(s)ds = -J_2(s)-2\int (s^{-1}J_1(s))'ds \\ &= -J_2(s)-2\frac{J_1(s)}{s}+C\end{aligned}$$

Note that since $\lim_{s\to 0}\frac{J_1(s)}{s}=\frac{1}{2}$ then

$$\int_0^x J_3(s)ds = \left[-J_2(s)-2\frac{J_1(s)}{s}\right]_0^x = 1-J_2(x)-2\frac{J_1(x)}{x}.$$

Exercise 16. Use the expansion of $\cos(x \sin\theta)$ involved in the proof of Proposition 3 to show that

$$\cos x = J_0(x) + 2\sum_{j=1}^{\infty}(-1)^j J_{2j}(x)$$
$$\sin x = 2\sum_{j=0}^{\infty}(-1)^j J_{2j+1}(x)$$
$$1 = J_0(x) + 2\sum_{j=1}^{\infty} J_{2j}(x)$$

Exercise 17. Use the integral representation of $J_n(x)$ to show that

$$J_n'(x) = \frac{1}{\pi}\int_0^{\pi} \sin(n\theta - x\sin(\theta))\,\sin\theta\, d\theta$$