# FOURIER-BESSEL SERIES AND BOUNDARY VALUE PROBLEMS IN CYLINDRICAL COORDINATES

## 1. Exercises

In Exercises 1 to 9, find the $J_\alpha$-Bessel series of the given function $f(x)$ over the given interval and with respect to the given endpoint condition.

**Exercise 1.** $f(x) = 100$ over $[0,\ 5]$, and $J_0(z) = 0$.

For $x \in (0,\ 5)$ we have

$$100 = \sum_{j=1}^{\infty} c_j\, J_0(z_j x/5)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_0(z) = 0$ and

$$c_j = \frac{\langle 100,\ J_0(z_j x/5)\rangle_x}{\|J_0(z_j x/5)\|_x^2} = \frac{2}{25 J_1(z_j)^2} \int_0^5 100 J_0(z_j x/5)\, x\, dx\,.$$

We have

$$\int_0^5 100 J_0(z_j x/5)\, x\, dx = 100\frac{25}{z_j^2} \int_0^{z_j} t\, J_0(t)\, dt = 100\frac{25}{z_j^2}\,[tJ_1(t)]_0^{z_j} = 100\frac{25\, J_1(z_j)}{z_j}$$

Thus $c_j = \dfrac{200}{z_j J_1(z_j)}$ and

$$100 = 200 \sum_{j=1}^{\infty} \frac{J_0(z_j x/5)}{z_j J_1(z_j)}\,.$$

**Exercise 2.** $f(x) = x$ over $[0,\ 7]$, and $J_1(z) = 0$.

**Exercise 3.** $f(x) = -5$ over $[0,\ 1]$, and $J_2(z) = 0$.

For $x \in (0,\ 1)$ we have

$$-5 = \sum_{j=1}^{\infty} c_j\, J_2(z_j x)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_2(z) = 0$ and

$$c_j = \frac{\langle -5,\ J_2(z_j x)\rangle_x}{\|J_2(z_j x)\|_x^2} = \frac{2}{J_3(z_j)^2} \int_0^1 (-5)\, J_2(z_j x)\, x\, dx\,.$$

We have

$$\begin{aligned}\int tJ_2(t)dt &= \int t^2(t^{-1}J_2(t))dt = \int t^2(-t^{-1}J_1(t))'dt \\ &= -tJ_1(t) + 2\int J_1(t)dt = -tJ_1(t) + 2\int(-J_0(t))'dt \\ &= -tJ_1(t) - 2J_0(t) + C\end{aligned}$$

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$$\int_0^1 J_2(z_j x)\, x\, dx \;= \frac{1}{z_j^2} \int_0^{z_j} t\, J_2(t)\, dt = \frac{1}{z_j^2} \left[-tJ_1(t) - 2J_0(t)\right]_0^{z_j}$$

$$= \frac{2 - 2J_0(z_j) - z_j J_1(z_j)}{z_j^2}$$

Thus $c_j = 10\dfrac{2 - 2J_0(z_j) - z_j J_1(z_j)}{z_j^2 J_3(z_j)^2}$ and

$$-5 = 10 \sum_{j=1}^{\infty} \frac{2 - 2J_0(z_j) - z_j J_1(z_j)}{z_j^2 J_3(z_j)^2}\, J_2(z_j x)\,.$$

**Exercise 4.** $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases}$ over $[0,\ 2]$, and $J_0(z) = 0$.

**Exercise 5.** $f(x) = x$ over $[0,\ 3]$, and $J_1'(z) = 0$

For $x \in\ (0,\ 3)$ we have

$$x = \sum_{j=1}^{\infty} c_j\, J_1(z_j x/3)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_1'(z) = 0$ and

$$c_j = \frac{\langle x,\ J_1(z_j x/3)\rangle_x}{\|J_1(z_j x/3)\|_x^2} = \ .$$

This time $\|J_1(z_j x/3)\|_x^2 = \dfrac{9(z_j^2 - 1)}{2z_j^2}\, J_1(z_j)^2$. We have

$$\langle x,\ J_1(z_j x/3)\rangle_x = \int_0^3 x^2 J_1(z_j x/3)\, x\, dx \;= \frac{27}{z_j^3} \int_0^{z_j} t^2\, J_1(t)\, dt = \frac{27}{z_j^3}\, z_j^2 J_1(z_j) = \frac{27\, J_1(z_j)}{z_j}$$

Thus $c_j = \dfrac{6z_j}{(z_j^2 - 1)J_1(z_j)}$ and

$$x = 6 \sum_{j=1}^{\infty} \frac{6z_j}{(z_j^2 - 1)J_1(z_j)}\, J_1(z_j x/3)\,.$$

**Exercise 6.** $f(x) = 1$ over $[0,\ 3]$, and $J_0(z) + zJ_0'(z) = 0$.

**Exercise 7.** $f(x) = x^2$ over $[0,\ 3]$, and $J_0(z) = 0$ (leave the coefficients in an integral form).

For $x \in\ (0,\ 3)$ we have

$$x^2 = \sum_{j=1}^{\infty} c_j\, J_0(z_j x/3)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_0(z) = 0$ and

$$c_j = \frac{\langle x^2,\ J_0(z_j x/3)\rangle_x}{\|J_0(z_j x/3)\|_x^2} = \frac{2}{9J_1(z_j)^2} \int_0^3 100 J_0(z_j x/3)\, x\, dx \;= \frac{18}{z_j^4 J_1(z_j)^2} \int_0^{z_j} t^3 J_0(t)\, dt.$$

We have

$$x^2 = \sum_{j=1}^{\infty} \left[ \frac{18}{z_j^4 J_1(z_j)^2} \int_0^{z_j} t^3 J_0(t)\, dt \right] J_0(z_j x/3)\,.$$

**Exercise 8.** $f(x) = x^2$ over $[0,\ 3]$, and $J_3(z) = 0$ (leave the coefficients in an integral form).

**Exercise 9.** $f(x) = \sqrt{x}$ over $[0,\ \pi]$, and $J_{1/2}(z) = 0$ (use the explicit expression of $J_{1/2}$ and relate to Fourier series).

For $x \in (0,\ \pi)$, we have

$$\sqrt{x} = \sum_{j=1}^{\infty} c_j J_{1/2}(z_j x/\pi)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_{1/2}(z) = 0$ and

$$c_j = \frac{\langle \sqrt{x},\ J_{1/2}(z_j x/\pi)\rangle_x}{\left\|J_{1/2}(z_j x/\pi)\right\|_x^2} = \frac{2}{\pi^2 J_{3/2}(z_j)^2}\int_0^{\pi} \sqrt{x} J_{1/2}(z_j x/\pi)\, x\, dx\,.$$

Now we use the expressions

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{3/2}(t) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

to get $z_j = j\pi,\ \ J_{3/2}(j\pi) = \dfrac{(-1)^{j+1}\sqrt{2}}{\pi\sqrt{j}}$, and

$$\int_0^{\pi} \sqrt{x} J_{1/2}(z_j x/\pi)\, x\, dx = \int_0^{\pi} \sqrt{x}\sqrt{\frac{2}{\pi j x}}\,\sin(jx)\, x dx = \frac{2\pi(-1)^{j+1}}{j\sqrt{j}}$$

Therefore $c_j = \dfrac{\sqrt{2\pi}\,(-1)^{j+1}}{\sqrt{j}}$ and we have the expansion

$$\sqrt{x} = \sum_{j=1}^{\infty} c_j J_0(z_j x/\pi) = \sum_{j=1}^{\infty} \frac{\sqrt{2\pi}\,(-1)^{j+1}}{\sqrt{j}}\sqrt{\frac{2}{\pi j x}}\,\sin(jx)$$

This expansion is equivalent (after multiplying by $\sqrt{x}$) to $x = 2\displaystyle\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sin(jx)$ which the Fourier sine expansion of $x$ over the interval $[0,\ \pi]$.

In exercises 11 to 14, solve the indicated boundary value problem that deal with heat flow in a circular domain.

**Exercise 11.**

$$\begin{array}{ll} u_t = 2\left(u_{rr} + \dfrac{u_r}{r}\right) & 0 < r < 2,\ t > 0, \\ u(2,t) = 0 & t > 0, \\ u(r,0) = 5 & 0 < r < 2. \end{array}$$

If $u(r,t) = R(r)T(t)$ solves the homogeneous part of the BVP, then $R$ and $T$ satisfy the ODE problems

$$\left\{\begin{array}{ll} r^2R''(r) + rR'(r) + \lambda r^2 R(r) = 0 & 0 < r < 2 \\ R(2) = 0 & \end{array}\right., \quad \text{and} \quad T'(t) + 2\lambda T(t) = 0, \quad t > 0\,.$$

The eigenvalues and eigenfunctions of the $R$-problem (singular SL-problem) are:

$$\lambda_j = \left(\frac{z_j}{2}\right)^2 \quad \text{and} \quad R_j(r) = J_0(z_j r/2)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_0(z) = 0$. The corresponding independent solution of the $T$-problem is $T_j(t) = \mathrm{e}^{-2z_j^2 t}$. The general solution has the series representation

$$u(r,t) = \sum_{j=1}^{\infty} c_j\,\mathrm{e}^{-2z_j^2 t}\, J_0(z_j r/2)\,.$$

The nonhomogeneous condition implies that

$$u(r,0) = 5 = \sum_{j=1}^{\infty} c_j\, J_0(z_j r/2)\,.$$

Hence

$$\begin{aligned} c_j &= \frac{\langle 5,\, J_0(z_j r/2)\rangle_r}{\|J_0(z_j r/2)\|_r^2} = \frac{5}{2J_1(z_j)^2}\int_0^2 rJ_0(z_j r/2)dr = \frac{10}{J_1(z_j)^2 z_j^2}\int_0^{z_j} tJ_0(t)dt \\ &= \frac{10}{J_1(z_j)^2 z_j^2}\left[tJ_1(t)\right]_0^{z_j} = \frac{10}{J_1(z_j) z_j} \end{aligned}$$

The solution of the BVP is

$$u(r,t) = 10\sum_{j=1}^{\infty} \frac{1}{J_1(z_j) z_j} \mathrm{e}^{-2z_j^2 t}\, J_0(z_j r/2)\,.$$

**Exercise 12.**

$$\begin{array}{ll} u_t = 2\left(u_{rr} + \dfrac{u_r}{r}\right) & 0 < r < 2,\ t > 0, \\ u_r(2,t) = 0 & t > 0, \\ u(r,0) = 5 & 0 < r < 2. \end{array}$$

**Exercise 13.**

$$\begin{array}{ll} u_t = 2\left(u_{rr} + \dfrac{u_r}{r}\right) & 0 < r < 2,\ t > 0, \\ 2u(2,t) - u_r(2,t) = 0 & t > 0, \\ u(r,0) = 5 & 0 < r < 2. \end{array}$$

If $u(r,t) = R(r)T(t)$ solves the homogeneous part of the BVP, then $R$ and $T$ satisfy the ODE problems

$$\left\{\begin{array}{ll} r^2R''(r) + rR'(r) + \lambda r^2 R(r) = 0 & 0 < r < 2 \\ R(2) = 0 & \end{array}\right. , \quad \text{and} \quad T'(t) + 2\lambda T(t) = 0, \quad t > 0\,.$$

The eigenvalues and eigenfunctions of the $R$-problem (singular SL-problem) are:

$$\lambda_j = \left(\frac{z_j}{2}\right)^2 \quad \text{and} \quad R_j(r) = J_0(z_j r/2)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $-2J_0(z) + zJ_0'(z) = 0$. This time the norms of the eigenfunctions satisfy

$$\|J_0(z_j r/2)\|_r^2 = \frac{2(z_j^2 - 4)}{z_j^2} J_0(z_j)^2\,.$$

The corresponding independent solution of the $T$-problem is $T_j(t) = \mathrm{e}^{-2z_j^2 t}$. The general solution has the series representation

$$u(r,t) = \sum_{j=1}^{\infty} c_j\, \mathrm{e}^{-2z_j^2 t}\, J_0(z_j r/2)\,.$$

The nonhomogeneous condition implies that

$$u(r,0) = 5 = \sum_{j=1}^{\infty} c_j\, J_0(z_j r/2)\,.$$

Hence

$$\begin{aligned} c_j &= \frac{\langle 5,\, J_0(z_j r/2)\rangle_r}{\|J_0(z_j r/2)\|_r^2} = \frac{5z_j^2}{2(z_j^2-4)J_0(z_j)^2}\int_0^2 r J_0(z_j r/2)dr \\ &= \frac{10}{(z_j^2-4)J_0(z_j)^2}\int_0^{z_j} tJ_0(t)dt = \frac{10}{(z_j^2-4)J_0(z_j)^2}\,[tJ_1(t)]_0^{z_j} \\ &= \frac{10\, z_j J_1(z_j)}{(z_j^2-4)J_0(z_j)^2}\,. \end{aligned}$$

The solution of the BVP is

$$u(r,t) = 10\sum_{j=1}^{\infty}\frac{z_j J_1(z_j)}{(z_j^2-4)J_0(z_j)^2}\,\mathrm{e}^{-2z_j^2 t}\,J_0(z_j r/2)\,.$$

**Exercise 14.** Find a solution $u$ of the form $u(r,\theta,t) = v(r,t)\sin(2\theta)$ of the problem.

$$\begin{array}{ll} u_t = 2\left(u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2}\right) & 0<r<2,\ 0\le\theta\le 2\pi,\ t>0, \\ u(2,\theta,t) = 0 & 0\le\theta\le 2\pi,\ t>0, \\ u(r,\theta,0) = 5r^2\sin(2\theta) & 0<r<2,\ 0\le\theta\le 2\pi,\ . \end{array}$$

**Exercise 15.**

$$\begin{array}{ll} u_t = \left(u_{rr} + \dfrac{u_r}{r} - \dfrac{9u}{r^2}\right) & 0<r<1,\ t>0, \\ u(1,t) = 0 & t>0, \\ u(r,0) = r^3 & 0<r<1. \end{array}$$

If $u(r,t) = R(r)T(t)$ solves the homogeneous part of the BVP, then $R$ and $T$ satisfy the ODE problems

$$\begin{cases} r^2R''(r) + rR'(r) + (\lambda r^2 - 9)R(r) = 0 & 0<r<1 \\ R(1) = 0 \end{cases}, \quad \text{and} \quad T'(t) + \lambda T(t) = 0,\quad t>0\,.$$

The eigenvalues and eigenfunctions of the $R$-problem (singular SL-problem) are:

$$\lambda_j = z_j^2 \quad \text{and} \quad R_j(r) = J_3(z_j r)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_3(z) = 0$. The corresponding independent solution of the $T$-problem is $T_j(t) = \mathrm{e}^{-z_j^2 t}$. The general solution has the series representation

$$u(r,t) = \sum_{j=1}^{\infty} c_j\,\mathrm{e}^{-z_j^2 t}\,J_3(z_j r)\,.$$

The nonhomogeneous condition implies that

$$u(r,0) = r^3 = \sum_{j=1}^{\infty} c_j\,J_3(z_j r)\,.$$

Hence

$$\begin{aligned} c_j &= \frac{\langle r^3,\, J_3(z_j r)\rangle_r}{\|J_3(z_j r)\|_r^2} = \frac{2}{J_4(z_j)^2}\int_0^1 r^4 J_3(z_j r)dr = \frac{2}{J_4(z_j)^2 z_j^5}\int_0^{z_j} t^4 J_3(t)dt \\ &= \frac{2}{J_4(z_j)^2 z_j^5}\left[t^4 J_4(t)\right]_0^{z_j} = \frac{2}{J_4(z_j) z_j} \end{aligned}$$

The solution of the BVP is

$$u(r,t) = 2\sum_{j=1}^{\infty} \frac{1}{z_j J_4(z_j) z_j} \mathrm{e}^{-z_j^2 t}\, J_3(z_j r)\,.$$

In exercises 16 to 19, solve the indicated boundary value problem that deal with wave propagation in a circular domain.

**Exercise 16.**

$$\begin{array}{ll} u_{tt} = 2\left(u_{rr} + \dfrac{u_r}{r}\right) & 0<r<3,\ t>0, \\ u(3,t) = 0 & t>0, \\ u(r,0) = 9 - r^2 & 0<r<3, \\ u_t(r,0) = 0 & 0<r<3. \end{array}$$

If $u(r,t) = R(r)T(t)$ solves the homogeneous part of the BVP, then $R$ and $T$ satisfy the ODE problems

$$\left\{ \begin{array}{l} r^2R''(r) + rR'(r) + \lambda r^2 R(r) = 0 \quad 0<r<3 \\ R(3) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} T''(t) + 2\lambda T(t) = 0, \quad t>0\,, \\ T'(0) = 0 \end{array} \right.$$

The eigenvalues and eigenfunctions of the $R$-problem (singular SL-problem) are:

$$\lambda_j = \left(\frac{z_j}{3}\right)^2 \quad \text{and} \quad R_j(r) = J_0(z_j r/3)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_0(z) = 0$. The corresponding independent solution of the $T$-problem is $T_j(t) = \cos\left(\sqrt{2}\, z_j t/3\right)$. The general solution has the series representation

$$u(r,t) = \sum_{j=1}^{\infty} c_j \,\cos\left(\sqrt{2}\, z_j t/3\right)\, J_0(z_j r/3)\,.$$

The nonhomogeneous condition implies that

$$u(r,0) = 9 - r^2 = \sum_{j=1}^{\infty} c_j\, J_0(z_j r/3)\,.$$

Hence

$$\begin{aligned} c_j &= \frac{\langle 9 - r^2,\, J_0(z_j r/3)\rangle_r}{\|J_0(z_j r/3)\|_r^2} \\ &= \frac{2}{9J_1(z_j)^2} \int_0^3 (9-r^2) r J_0(z_j r/3)\, dr\,. \end{aligned}$$

We have

$$\begin{aligned} \int_0^3 (9-r^2) r J_0(z_j r/3)\, dr &= 9\int_0^3 r J_0(z_j r/3)\, dr - \int_0^3 r^2 r J_0(z_j r/3)\, dr \\ &= \frac{81}{z_j^2} \int_0^{z_j} t J_0(t)\, dt - \frac{81}{z_j^4} \int_0^{z_j} t^3 J_0(t)\, dt \end{aligned}$$

Since

$$\begin{aligned}\int tJ_0(t)\,dt &= tJ_1(t)+C\\ \int t^3J_0(t)\,dt &= \int t^2(tJ_1(t))'\,dt = t^3J_1(t)-2\int t^2J_1(t)\,dt\\ &= t^3J_1(t)-2t^2J_2(t)+C\end{aligned}$$

then

$$\begin{aligned}\int_0^3(9-r^2)rJ_0(z_jr/3)\,dr &= \frac{81}{z_j^2}\left[tJ_1(t)\right]_0^{z_j}-\frac{81}{z_j^4}\left[t^3J_1(t)-2t^2J_2(t)\right]_0^{z_j}\\ &= \frac{162\,J_2(z_j)}{z_j^2}.\end{aligned}$$

It follows that $c_j=\dfrac{36\,J_2(z_j)}{z_j^2J_1(z_j)^2}$ and the solution of the BVP is

$$u(r,t)=36\sum_{j=1}^{\infty}\frac{J_2(z_j)}{z_j^2J_1(z_j)^2}\cos\left(\sqrt{2}\,z_jt/3\right)\,J_0(z_jr/3)\,.$$

**Exercise 17.**

$$\begin{array}{ll} u_{tt}=\left(u_{rr}+\dfrac{u_r}{r}\right) & 0<r<1,\ t>0,\\ u(1,t)=0 & t>0,\\ u(r,0)=0 & 0<r<1,\\ u_t(r,0)=g(r) & 0<r<1.\end{array}$$

where

$$g(r)=\begin{cases}1 & \text{if } 0<r<1/2,\\ 0 & \text{if } (1/2)<r<1.\end{cases}$$

**Exercise 18.** Find a solution $u$ of the form $u(r,\theta,t)=v(r,t)\cos\theta$

$$\begin{cases} u_{tt}=\left(u_{rr}+\dfrac{u_r}{r}+\dfrac{u_{\theta\theta}}{r^2}\right) & 0<r<1,\ t>0,\ 0\le\theta\le2\pi\\ u(1,\theta,t)=0 & t>0,\ 0\le\theta\le2\pi\\ u(r,\theta,0)=0 & 0<r<1,\ 0\le\theta\le2\pi\\ u_t(r,\theta,0)=g(r)\cos\theta & 0<r<1,\ 0\le\theta\le2\pi.\end{cases}$$

where

$$g(r)=\begin{cases}r & \text{if } 0<r<1/2,\\ 0 & \text{if } (1/2)<r<1.\end{cases}$$

In order for $u(r,\theta,t)=v(r,t)\cos\theta$ to solve the BVP, the function $v(r,t)$ needs to solve the following BVP

$$(*)\quad\begin{cases} v_{tt}=v_{rr}+\dfrac{v_r}{r}-\dfrac{v}{r^2} & 0<r<1,\ t>0,\\ v(1,t)=0 & t>0,\\ v(r,0)=0 & 0<r<1,\\ v_t(r,0)=g(r) & 0<r<1.\end{cases}$$

If $v(r,t)=R(r)T(t)$ solves the homogeneous part of BVP $(*)$, then $R$ and $T$ satisfy the ODE problems

$$\begin{cases} r^2R''(r)+rR'(r)+(\lambda r^2-1)R(r)=0 & 0<r<1\\ R(1)=0\end{cases}\quad\text{and}\quad\begin{cases} T''(t)+\lambda T(t)=0, & t>0\,,\\ T(0)=0\end{cases}$$

The eigenvalues and eigenfunctions of the $R$-problem (singular SL-problem) are:

$$\lambda_j = z_j^2 \quad \text{and} \quad R_j(r) = J_1(z_j r)\,,$$

where $z_j$ is the $j^{\text{th}}$ positive root of $J_1(z) = 0$. The corresponding independent solution of the $T$-problem is $T_j(t) = \sin(z_j t)$. The general solution has the series representation

$$v(r,t) = \sum_{j=1}^{\infty} c_j \,\sin(z_j t)\, J_1(z_j r).$$

We have

$$v_t(r,t) = \sum_{j=1}^{\infty} c_j \, z_j \,\cos(z_j t)\, J_1(z_j r).$$

The nonhomogeneous condition implies that

$$v_t(r,0) = g(r) = \sum_{j=1}^{\infty} c_j \, z_j \, J_1(z_j r)\,.$$

Hence

$$c_j \, z_j = \frac{\langle g(r),\, J_1(z_j r)\rangle_r}{\|J_1(z_j r)\|_r^2} = \frac{2}{J_2(z_j)^2} \int_0^1 g(r) r J_1(z_j r)\, dr\,.$$

We have

$$\begin{aligned}\int_0^1 g(r) r J_1(z_j r)\, dr \;&= \int_0^{1/2} r^2 J_1(z_j r)\, dr = \frac{1}{z_j^3}\int_0^{z_j/2} t^2 J_1(t)\, dt \\ &= \frac{1}{z_j^3}\left[t^2 J_2(t)\right]_0^{z_j/2} = \frac{J_2(z_j/2)}{4 z_j}\end{aligned}$$

It follows that $c_j z_j = \dfrac{J_2(z_j/2)}{2 z_j J_2(z_j)^2}$ and the solution of the BVP $(*)$ is

$$v(r,t) = \sum_{j=1}^{\infty} \frac{J_2(z_j/2)}{2 z_j^2 J_2(z_j)^2} \,\sin(z_j t)\, J_1(z_j r).$$

The solution $u$ of the original BVP is

$$u(r,\theta,t) = v(r,t)\cos\theta = \sum_{j=1}^{\infty} \frac{J_2(z_j/2)}{2 z_j^2 J_2(z_j)^2} \,\sin(z_j t)\, J_1(z_j r)\, \cos\theta\,.$$

**Exercise 19.** Find a solution $u$ of the form $u(r,\theta,t) = v(r,t)\cos\theta$

$$\begin{array}{ll} u_{tt} = \left(u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2}\right) & 0 < r < 1,\; t > 0, \\ u(1,t) = 0 & t > 0, \\ u(r,0) = J_1(z_{1,1} r)\sin\theta & 0 < r < 1, \\ u_t(r,0) = g(r)\cos\theta & 0 < r < 1. \end{array}$$

where $z_{1,1}$ is the first positive zero of $J_1$ and where

$$g(r) = \begin{cases} r & \text{if } 0 < r < 1/2, \\ 0 & \text{if } (1/2) < r < 1. \end{cases}$$

In exercises 20 to 23, solve the Helmholtz equation in the disk with radius $L$

**Exercise 20.** $L = 2$

$$\Delta u - u = 2, \quad u(2,\theta) = 0$$

We construct a series solution in terms of the eigenfunctions of the associated 2-dimensional Helmholtz eigenvalue problem

$$\begin{cases} \Delta u(r,\theta) + \lambda u(r,\theta) = 0 & 0 < r < 2, \quad 0 \le \theta \le 2\pi\,, \\ u(2,\theta) = 0 & 0 \le \theta \le 2\pi\,. \end{cases}$$

The eigenvalues and eigenfunctions are:

$$\lambda_{j,m} = \left(\frac{z_{j,m}}{2}\right)^2, \qquad \begin{cases} u^1_{j,m}(r,\theta) = \cos(m\theta) J_m\left(\frac{z_{j,m} r}{2}\right), \\ u^2_{j,m}(r,\theta) = \sin(m\theta) J_m\left(\frac{z_{j,m} r}{2}\right), \end{cases}$$

where $z_{j,m}$ is the $j$-th positive roots of the equation $J_m(z) = 0$.

We expand $F(r,\theta) = 2$ and $u(r,\theta)$ into these eigenfunctions (Fourier-Bessel series). We seek then a solution $u$ of the form

$$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{j=1}^{\infty} \left[A_{m,j}\cos(m\theta) + B_{m,j}\sin(m\theta)\right] J_m\left(\frac{z_{j,m} r}{2}\right)$$

By using $\Delta u^{1,2}_{m,j} = -\lambda_{m,j} u^{1,2}_{m,j}$, we deduce that

$$\Delta u - u = -\sum_{m=0}^{\infty}\sum_{j=1}^{\infty} (1+\lambda_{m,j}) \left[A_{m,j}\cos(m\theta) + B_{m,j}\sin(m\theta)\right] J_m\left(\frac{z_{j,m} r}{2}\right)$$

The Fourier-Bessel expansion of $F(r,\theta) = 2$ is

$$2 = \sum_{j=1}^{\infty} \frac{4}{z_{j,0} J_1(z_{j,0})} J_0\left(\frac{z_{j,0} r}{2}\right).$$

It follows after identifying the series representation of $\Delta u - u$ and 2 that

$$A_{m,j} = B_{m,j} = 0 \quad \text{for } m > 0$$

$$A_{0,j} = \frac{-4}{z_{j,0}(1+\lambda_{0,j}) J_1(z_{j,0})} = \frac{-16}{z_{j,0}(4 + z_{0,j}^2) J_1(z_{j,0})}.$$

The solution of the BVP is

$$u(r,\theta) = -16 \sum_{j=1}^{\infty} \frac{1}{z_{j,0}(4 + z_{0,j}^2) J_1(z_{j,0})} J_0\left(\frac{z_{j,0} r}{2}\right)$$

**Exercise 21.** $L = 1$

$$\Delta u = r\sin\theta, \quad u(1,\theta) = 0$$

**Exercise 22.** $L = 3$

$$\Delta u + 2u = -1 + 5r^3\cos(3\theta), \quad u(3,\theta) = 0$$

We construct a series solution in terms of the eigenfunctions of the associated 2-dimensional Helmholtz eigenvalue problem

$$\begin{cases} \Delta u(r,\theta) + \lambda u(r,\theta) = 0 & 0 < r < 3, \quad 0 \le \theta \le 2\pi\,, \\ u(3,\theta) = 0 & 0 \le \theta \le 2\pi\,. \end{cases}$$

The eigenvalues and eigenfunctions are:

$$\lambda_{j,m} = \left(\frac{z_{j,m}}{3}\right)^2, \qquad \begin{cases} u^1_{j,m}(r,\theta) = \cos(m\theta) J_m\left(\frac{z_{j,m} r}{3}\right), \\ u^2_{j,m}(r,\theta) = \sin(m\theta) J_m\left(\frac{z_{j,m} r}{3}\right), \end{cases}$$

where $z_{j,m}$ is the $j$-th positive roots of the equation $J_m(z) = 0$.

We expand $F(r,\theta) = -1 + 5r^3\cos(3\theta)$ and $u(r,\theta)$ into these eigenfunctions (Fourier-Bessel series). We seek then a solution $u$ of the form

$$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\left[A_{m,j}\cos(m\theta) + B_{m,j}\sin(m\theta)\right] J_m\left(\frac{z_{j,m}r}{3}\right)$$

By using $\Delta u_{m,j}^{1,2} = -\lambda_{m,j}u_{m,j}^{1,2}$, we deduce that

$$\Delta u + 2u = \sum_{m=0}^{\infty}\sum_{j=1}^{\infty}(2-\lambda_{m,j})\left[A_{m,j}\cos(m\theta) + B_{m,j}\sin(m\theta)\right] J_m\left(\frac{z_{j,m}r}{2}\right)$$

The Fourier-Bessel expansion of the functions 1 and $r^3\cos(3\theta)$ are:

$$1 = \sum_{j=1}^{\infty}\frac{2}{z_{j,0}J_1(z_{j,0})}\,J_0\left(\frac{z_{j,0}r}{3}\right),$$

$$r^3\cos(3\theta) = 54\sum_{j=1}^{\infty}\frac{1}{z_{j,3}J_4(z_{j,3})}\,\cos(3\theta)\,J_3\left(\frac{z_{j,3}r}{3}\right)$$

We have then

$$-1 + 5r^3\cos(3\theta) = -\sum_{j=1}^{\infty}\frac{2}{z_{j,0}J_1(z_{j,0})}\,J_0\left(\frac{z_{j,0}r}{3}\right) + 270\sum_{j=1}^{\infty}\frac{1}{z_{j,3}J_4(z_{j,3})}\,\cos(3\theta)\,J_3\left(\frac{z_{j,3}r}{3}\right)$$

It follows after identifying the series representation of $\Delta u + 2u$ and $-1 + 5r^3\cos(3\theta)$ that

$$B_{m,j} = 0 \ \text{ for all } m;\ A_{m,j} = 0 \quad \text{for } m \neq 0 \text{ or } 3$$

$$A_{0,j} = \frac{-2}{z_{j,0}(2-\lambda_{0,j})J_1(z_{j,0})} = \frac{-18}{z_{j,0}(18-z_{0,j}^2)J_1(z_{j,0})}$$

$$A_{3,j} = \frac{270}{z_{j,3}(2-\lambda_{3,j})J_4(z_{j,3})} = \frac{2520}{z_{j,3}(18-z_{3,j}^2)J_4(z_{j,3})}$$

The solution of the BVP is

$$u(r,\theta) = \sum_{j=1}^{\infty}\frac{-18}{z_{j,0}(18-z_{0,j}^2)J_1(z_{j,0})}J_0\left(\frac{z_{j,0}r}{3}\right) + \sum_{j=1}^{\infty}\frac{2520}{z_{j,3}(18-z_{3,j}^2)J_4(z_{j,3})}\,\cos(3\theta)\,J_3\left(\frac{z_{j,3}r}{3}\right)$$

**Exercise 23.** Solve the following Dirichlet problem in the cylinder with radius 10 and height 20

$$\begin{array}{ll} u_{rr} + \dfrac{u_r}{r} + u_{zz} = 0 & 0 < r < 10,\ 0 < z < 20, \\ u(10,z) = 0 & 0 < z < 20, \\ u(r,0) = 0 & 0 < r < 10, \\ u(r,20) = 1 & 0 < r < 10. \end{array}$$